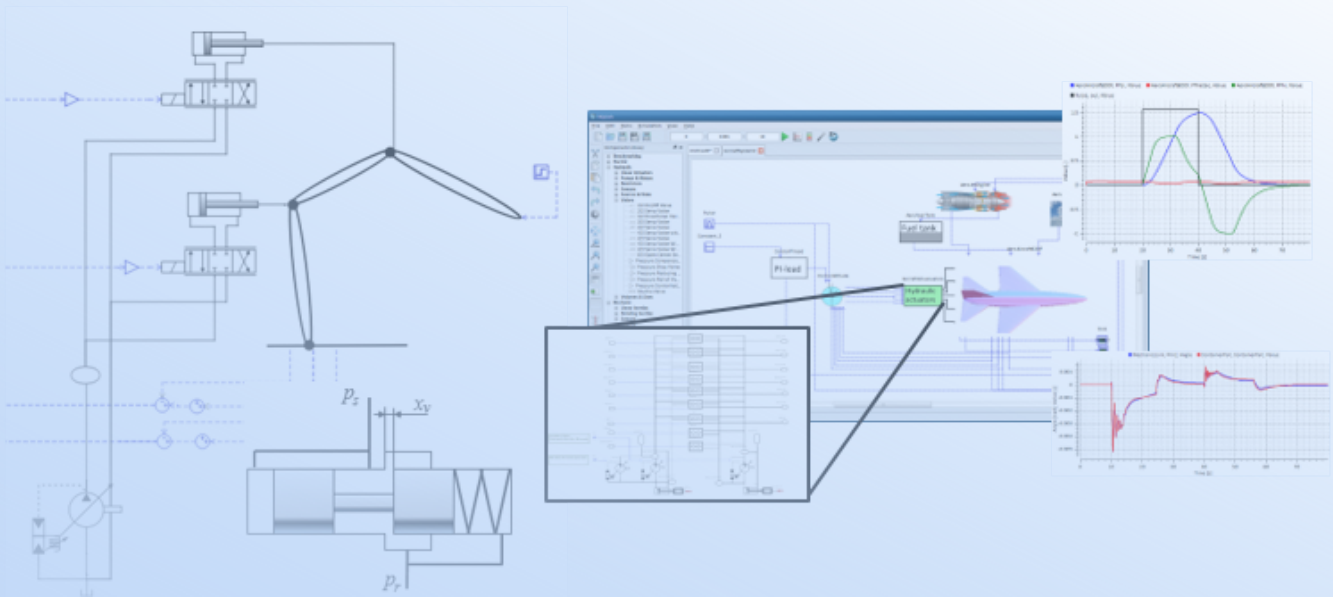


Transmission Line Modelling for System Simulation

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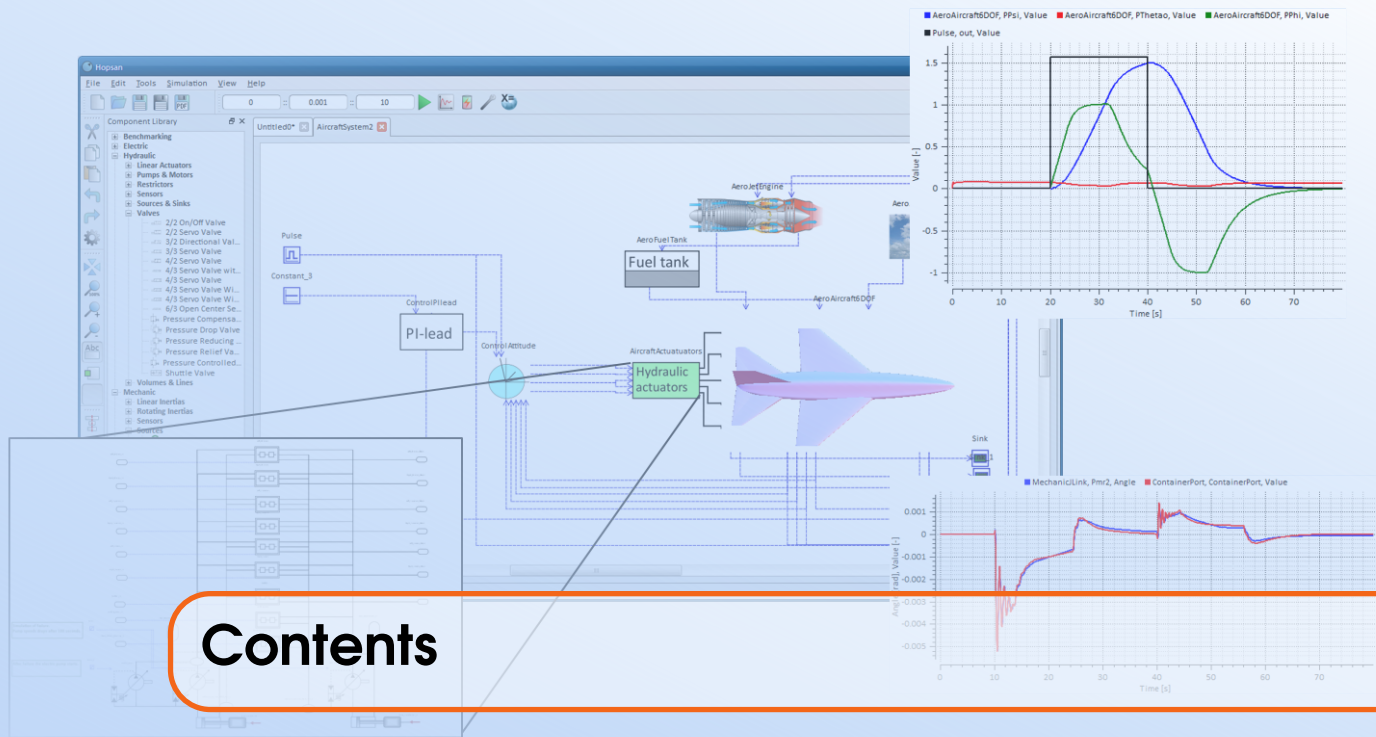
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Preface

This material is intended for students at an advance level with basic understanding in system dynamics and fluid power. It is a work in progress. The intention is to provide an understanding of system modelling and also the capability to create models at a detailed equation level.

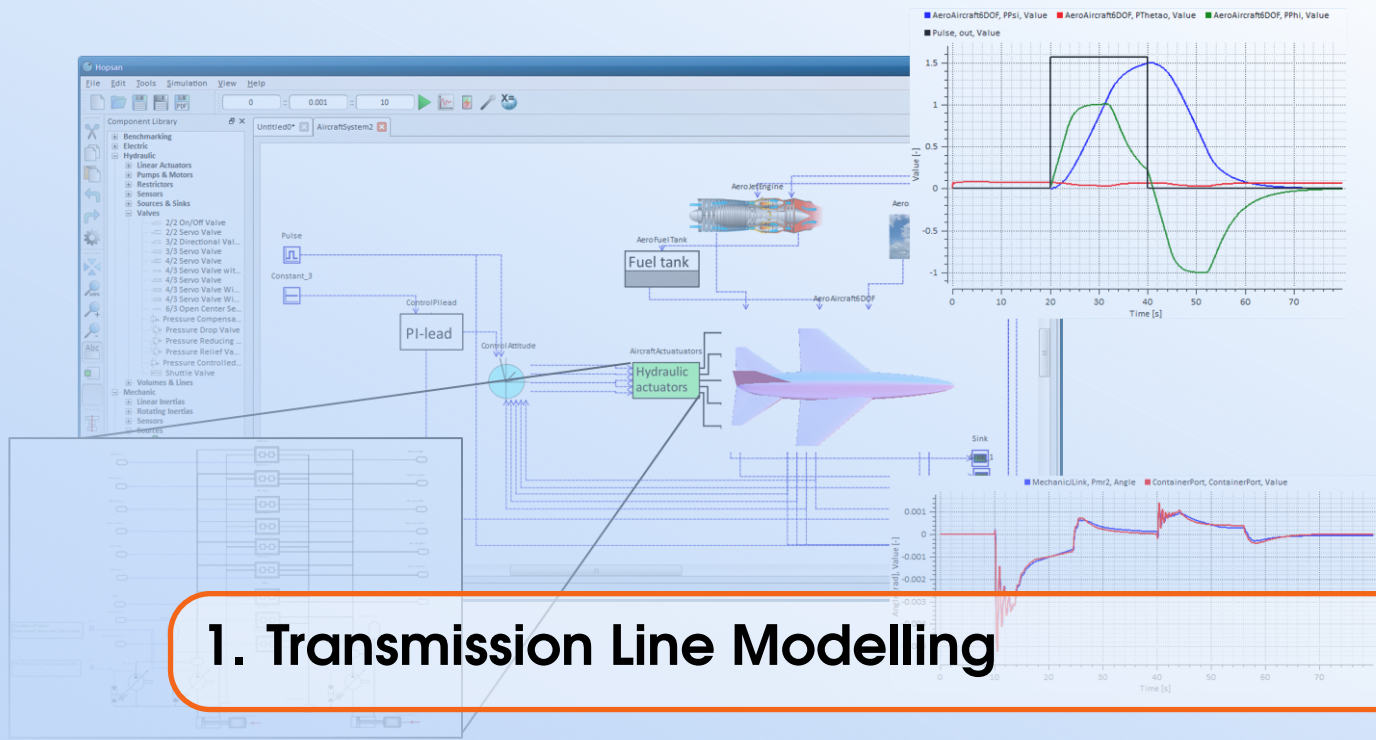
This material is about system simulation using transmission line dynamics to obtain partitioned models that can be used with local solvers, and can be used for parallel processing. Relying on local solvers also means that good scaling properties are obtained.



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Simulation of fluid power systems are characterized by difficulties such as very strong non-linearities, stiff differential equations and a high degree of complexity. Using conventional integration techniques it is often necessary to use very small time steps in order to be able to deal with numerically stiff problems, and strong non-linearities. A very suitable method for modelling and simulation of large complex dynamic systems is represented by distributed modelling using transmission line elements. The origin of this concept goes back at least to Auslander 1968 [1] who first introduced transmission lines (or bi-lateral delay lines). This method evolves naturally for calculation of pressures when pipelines are modelled with distributed parameters. This approach was adopted for simulation of fluid power systems with long lines in the HYTRAN program already in the seventies. A related method is the transmission line modelling method (TLM), or transmission line matrix method, presented by Johns and O'Brien [7] for simulation of electrical networks. Bi-directional delay lines was introduced in the Hopsan simulation package in [13] where it was also introduced for parallel simulation of systems. Since then it has also been used for real time simulation [11]. Multi thread simulation was introduced in [3].

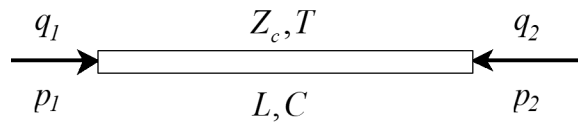


Figure 1.1: Transmission Line with Time delay T.

The hydraulic transmission line is an object with a complex dynamic behaviour. This behaviour can be investigated by solving the Navier Stokes equations for one dimensional unsteady flow. This is done e.g. in D'Souza and Oldenburger [15], and also in Ham [6] and Viersma [16]. However, for the case of flow without friction a much simpler approach can be made. In many hydraulic systems this is negligible and the line may be accurately described by its capacitance and a time delay for all signals passing through the line. The following list of properties is sufficient to describe such a line and to uniquely define a set of equations that can model the line.

- There is a time delay T for all information propagating through the line. (Flow speed is very small compared to the speed of sound in the line).
- Symmetry.
- Flows entering the line will produce an increase in pressure in steady state.
- Linear dynamics.

The first and second property yields

$$p_1(t) = f(q_1(t), q_2(t-T), p_2(t-T)) \quad (1.1)$$

$$p_2(t) = f(q_2(t), q_1(t-T), p_1(t-T)) \quad (1.2)$$

The third and fourth property yields

$$p_1(t) = Z_c q_1(t) + Z_c q_2(t-T) + p_2(t-T) \quad (1.3)$$

$$p_2(t) = Z_c q_2(t) + Z_c q_1(t-T) + p_1(t-T) \quad (1.4)$$

If we assume a steady state initial condition where $q_1 = -q_2$, this also implies that $p_2 = p_1$, and subject the line to a sudden change in q_1 , using eq (1.3) the corresponding pressure change can be calculated as

$$p_1(t) - p_1(t-T) = Z_c [q_1(t) - q_1(t-T)] \quad (1.5)$$

or

$$\Delta p_1(t) = Z_c \Delta q_1(t) \quad (1.6)$$

Equation (1.6) is called Joukowski's equation and describes a phenomenon called "water hammer". If a valve is suddenly closed this will cause a pressure transient in the pipe up-streams from the valve. "Water hammer" can also cause severe pressure transients in return lines i.e when a valve is suddenly opened to release flow to a the return line.

In order to collect all information that propagates from one end of a line to the other, the variables c_1, c_2 are introduced as

$$c_1(t) = Z_c q_2(t-T) + p_2(t-T) \quad (1.7)$$

$$c_2(t) = Z_c q_1(t-T) + p_1(t-T) \quad (1.8)$$

c_1, c_2 are called the characteristics and they represents the waves traveling in each direction through the line.

In this way (1.3) and (1.4) can be rewritten as

$$p_1(t) = Z_c q_1(t) + c_1(t) \quad (1.9)$$

$$p_2(t) = Z_c q_2(t) + c_2(t) \quad (1.10)$$

The boundary condition at the ends of the line are described by adjacent components.

The flow is typically a function of the pressure.

$$q_i(t) = f(p_i(t)) \quad (1.11)$$

Eq (1.9) and equation (1.11) produce the following set of equations that are to be solved at each end of the line.

$$q_i(t) = f(p_i(t)) \quad (1.12)$$

$$p_i(t) = Z_c q_i(t) + c_i(t) \quad (1.13)$$

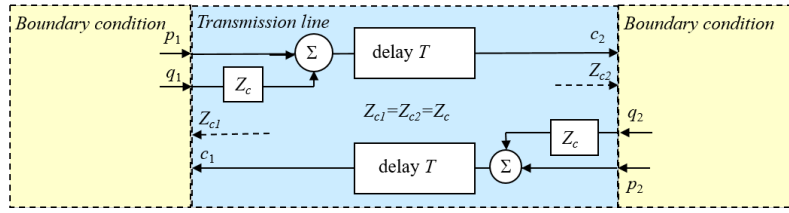


Figure 1.2: Block diagram of a lossless transmission line.

Using these equations together with equations (1.8) and (1.7) the block diagram in Fig 1.2 can be made.

The boundary blocks depends of course on the boundary condition at each end of the line. For the case of blocked end we have that $q_i(t) = 0$. Equation (1.13) then yields

$$p_i(t) = c_i(t) \quad (1.14)$$

For the case of an open end $p_i(t) = p_0$, where p_0 is a constant pressure. Equation (1.13) then yields

$$p_0 = Z_c q_i(t) + c_i(t) \quad (1.15)$$

Rearranging

$$q_i(t) = (p_0 - c_i(t))/Z_c \quad (1.16)$$

If we have a line with prescribed inlet flow and an open outlet (prescribed outlet pressure) as in Fig. 1.3 the block diagram for this system can be obtained by using (1.16) and (1.13) together with Fig. 1.2 to form the block diagram in Fig. 1.4.

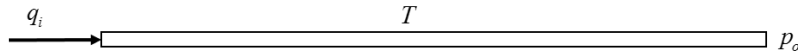


Figure 1.3: Transmission line with prescribed inlet flow and prescribed outlet pressure (open outlet)

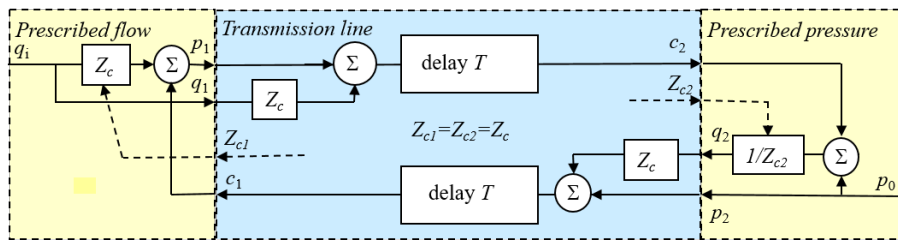


Figure 1.4: Block diagram of a transmission line with prescribed inlet flow and prescribed outlet pressure (open outlet).

Using the equations (1.7), (1.8), (1.9) and (1.13) the following explicit scheme is obtained:

$$c_1(t) = Z_c q_2(t - T) + p_2(t - T) \quad (1.17)$$

$$c_2(t) = Z_c q_1(t - T) + p_1(t - T) \quad (1.18)$$

$$p_1(t) = Z_c q_1(t) + c_1(t) \quad (1.19)$$

$$q_2(t) = (p_0 - c_2(t))/Z_c \quad (1.20)$$

Alternatively the equations can also be written as

$$c_1(t) = 2Z_c q_2(t - T) + c_2(t - T) \quad (1.21)$$

$$c_2(t) = 2Z_c q_1(t - T) + c_1(t - T) \quad (1.22)$$

$$p_1(t) = Z_c q_1(t) + c_1(t) \quad (1.23)$$

$$q_2(t) = (p_0 - c_2(t))/Z_c \quad (1.24)$$

This has the advantage that the interface becomes simpler since the pressure need not be transferred to the transmission line.

1.0.1 Hydraulic Capacitance

The existence of capacitance in the line is actually a consequence of the third initial assumption. That is, flows entering a line will produce an increase in pressure in steady state. The bulk modulus β of the oil is defined as:

$$\beta = -V \frac{dp}{dV} \quad (1.25)$$

where V is the oil volume

The hydraulic capacitance is defined as

$$\frac{dp}{dt} = \frac{q}{C} \quad (1.26)$$

but $q = -\frac{dV}{dt}$ which yields

$$\frac{dp}{dt} = -\frac{1}{C} \frac{dV}{dt} \quad (1.27)$$

Inverting (1.25) and multiplying both sides with $\frac{dp}{dt}$ yields

$$\frac{1}{\beta} \frac{dp}{dt} = -\frac{1}{V} \frac{dV}{dt} \quad (1.28)$$

Rearranging yields

$$\frac{dp}{dt} = -\frac{\beta}{V} \frac{dV}{dt} \quad (1.29)$$

Identification yields the capacitance C as

$$C = \frac{V}{\beta} = \frac{Al}{\beta} \quad (1.30)$$

The next step is to find out how this relates to the parameters T and Z_c of the line. If we have a line with a blocked outlet $q_2 = 0$. Equation (1.10) shifted one time step in (1.9) yields

$$p_1(t) = p_1(t - 2T) + Z_c [q_1(t) + q_1(t - 2T)] \quad (1.31)$$

Rearranging gives

$$\frac{p_1(t) - p_1(t - 2T)}{2T} = \frac{Z_c}{2T} [q_1(t) + q_1(t - 2T)] \quad (1.32)$$

For slow variations in p and q this can be written as

$$\frac{dp_1(t)}{dt} = \frac{Z_c}{T} q_1(t) \quad (1.33)$$

Identification with (1.27) yields

$$C = T/Z_c \quad (1.34)$$

It is interesting to note the similarity between (1.31) and the trapezoidal rule for integration

$$y(t) = y(t - h) + \frac{1}{h} [y'(t) + y'(t - h)] \quad (1.35)$$

They are identical if the time step h is replaced with $2T$ in the line. This means that if the trapezoidal rule is used for integrating pressure in a volume, this corresponds to the "exact" dynamics of a blocked lossless transmission line.

1.0.2 Hydraulic Inductance

Due to the inertia of the fluid, the hydraulic line will have some inductance. Newtons second law gives

$$F = m \frac{d^2x}{dt^2} \quad (1.36)$$

The hydraulic inductance L is defined as

$$\Delta p = L \frac{dq}{dt} \quad (1.37)$$

The mass of the fluid in the line can be written as

$$m = Al\rho \quad (1.38)$$

Here A is the area of the cross section of the line, l is the length and ρ the density of the fluid. Furthermore

$$F = A\Delta p \quad (1.39)$$

and

$$\frac{dq}{dt} = A \frac{d^2x}{dt^2} \quad (1.40)$$

This yields

$$\Delta p = \frac{l\rho}{A} \frac{dq}{dt} \quad (1.41)$$

Identification yields

$$L = \frac{l\rho}{A} \quad (1.42)$$

The next step is to find out how this relates to the parameters T and Z_c of the line.

Subtracting equation (1.4) from (1.3) yields

$$p_1(t) - p_2(t) + p_1(t - T) - p_2(t - T) = Z_c[q_1(t) - q_1(t - T) - (q_2(t) - q_2(t - T))] \quad (1.43)$$

Assuming a quasi steady state situation where p_1 and p_2 are constant and $q_1 = -q_2 = q$ equation (1.43) can be rewritten as:

$$2(p_1 - p_2) = 2Z_c[q(t) - q(t - T)] \quad (1.44)$$

near linear variations in q , $\Delta p = p_1 - p_2$ and dividing with $2T$ yields

$$\frac{\Delta p}{T} = Z_c \frac{\Delta q(t)}{dt} \quad (1.45)$$

For sufficiently small T this can be written as:

$$\frac{dp}{dt} = Z_c \frac{dq(t)}{dt} \quad (1.46)$$

Identification with (1.37) yields

$$L = Z_c T \quad (1.47)$$

Combining (1.47) and (1.47) the time delay T can be calculated.

$$\frac{L}{T} = \frac{T}{C} \quad (1.48)$$

This yields T as

$$T = \sqrt{LC} \quad (1.49)$$

Equation (1.49) in (1.47) yields

$$Z_c = \sqrt{\frac{L}{C}} \quad (1.50)$$

1.1 The Unit Transmission Line Element

When TLM is used as a technique for connecting components and subsystems as a general method the basic dynamic element is the unit transmission line. This is the case in the HOPSAN package where this is used to connect different components to each other. In the general case a unit transmission line element can be used to model both capacitances and inductances. In the HOPSAN-package, however, it is used mainly to represent capacitances (oil volumes and mechanical springs) except in electrical systems where it can be used also to represent inductances. In contrast to a general transmission line with an arbitrary delay time T , this is for the unit transmission line set to be equal to the time step h in the simulation, so that $T = h$. Here Z_c is the characteristic impedance of the line, p and q are pressures and flows respectively. Note that the main property of these equations is the time delay they introduce in the communication between the ends. The incident and reflective waves can also be viewed as an alternative set of variables. They represent the coordinate axes rotated 45 degrees relative a coordinate system based on p and $Z_c q$.

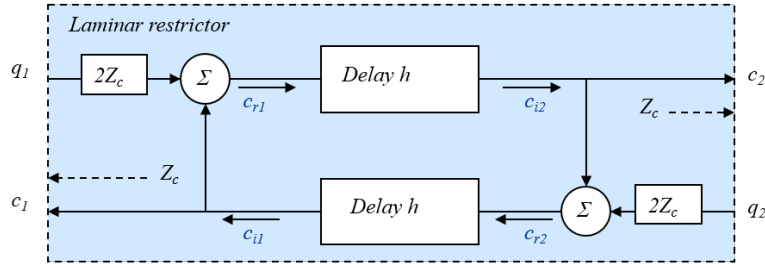


Figure 1.5: Block diagram of the unit transmission line element.

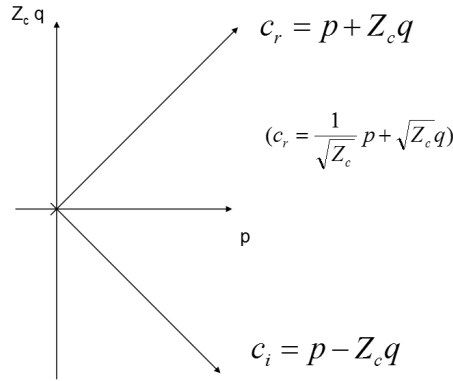


Figure 1.6: Relationship between the wave variables and flow and pressure.

Another interesting observation is found if p_2 in eq (1.3) is substituted with eq (1.4) (shifted T) and the outlet in 2 is blocked ($q_2 = 0$)

$$p_1(t + T) = p_1(t - T) + Z_c[q_1(t + T) + q_1(t - T)] \tag{1.51}$$

If this equation is compared to the trapezoidal method for integration eq (1.52)

$$y_{n+1} = y_n + \frac{h}{2} (F(y_{n+1}, t_{n+1}) + F(y_n, t_n)) \tag{1.52}$$

we see that they are the same if $T = h/2$

The implication of this is that if we use the trapezoidal method to integrate pressure in a volume (capacitance) between two components, this corresponds to introducing a short pipe instead of a pure capacitance see

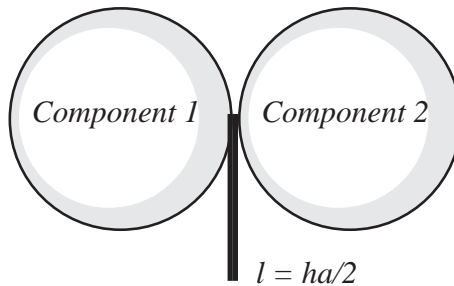


Figure 1.7: Modelling of capacitance using the trapezoidal method

If, however the volume is modelled as pipe to begin with this can be oriented so that it isolates the two components from each other and thereby isolate them numerically from each other since

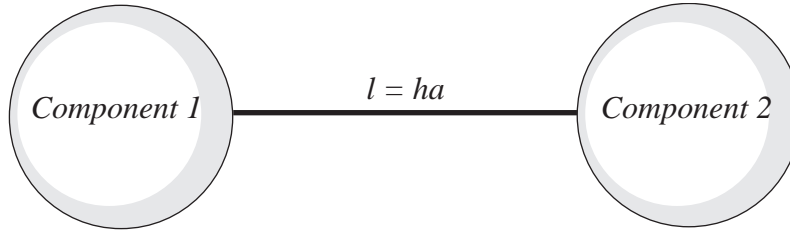


Figure 1.8: Modelling of capacitance using UTL-element.

there is a physically motivated time delay between the components. In order to represent a pure capacitance C with an UTL-element the length of the element will correspond to ha where h is the time step and a is the speed of sound in the fluid. The characteristic impedance is simply set to $Z_c = h/C$. For a more detailed discussion on the UTL-element as an integrator see ref [12] where also elements with more than two nodes are described. Introducing the characteristics c_1 and c_2 as:

$$c_2(t) = p_1(t) + Z_c q_1(t) \quad (1.53)$$

$$c_1(t) = p_2(t) + Z_c q_2(t) \quad (1.54)$$

Inserted in 1.3 and 1.4 yields

$$p_1(t+T) = c_1(t) + Z_c q_1(t+T) \quad (1.55)$$

$$p_2(t+T) = c_2(t) + Z_c q_2(t+T) \quad (1.56)$$

where c_1 and c_2 contain all information of the opposite node necessary for the calculation of pressure and flow at one node. At each component between the lines, we then have to solve the following system of equations.

$$q = q(p) \quad (1.57)$$

$$p = c + Z_c q \quad (1.58)$$

Here, q is a vector with all the flows in all the nodes connected to the component, p is the corresponding pressures, c the characteristics and finally Z_c is a diagonal matrix with the characteristic impedances in the diagonal. Limitation of the pressure due to cavitation is handled by the component. Compared with conventional modelling, where only eq (1.57) would be present (the pressures are then integrated in the adjacent volumes), an extra set of equations (1.58) has been added. These equations are, however, very simple and always the same.

Eqn (1.55) and (1.56) can also be used in (1.53) and (1.54) (shifted one time step)

$$c_2(t) = c_1(t-T) + 2Z_c q_1(t) \quad (1.59)$$

$$c_1(t) = c_2(t-T) + 2Z_c q_2(t) \quad (1.60)$$

An advantage with these expressions is that the wave variables are not manipulated by the component, and are consequently not affected by cavitation. As a result, the void is tracked and pressure cannot go above zero unless the void is replaced, which is a reasonable representation of what really is happening.

1.1.1 Volume with several connections

A very useful element is a joint of several transmission lines that can be used to represent a volume with an arbitrary number of connections. In this way it is possible to represent a pipeline system with a lumped capacitance, which is often a sufficiently accurate approximation. Note that each line element only has a delay of $T/2$. This is sufficient since there will always be a full time step

between two nodes. The characteristics from each node can be expressed as a function of old characteristics and flows.

$$c_i(t) = \frac{2}{N} \sum_{j=1}^N c_j(t-T) + 2Z_c q_j(t-T) - c_i(t-T) - 2Z_c q_i(t-T) \quad (1.61)$$

Alternatively, and more economical is to implement it as:

$$c_T(t) = \frac{2}{N} \sum_{j=1}^N c_j(t-T) + 2Z_c q_j(t-T) \quad (1.62)$$

$$c_i(t) = c_T(t) - c_i(t-T) - 2Z_c q_i(t-T) \quad (1.63)$$

Here the total volume has to be distributed to all the involved line elements. The characteristic impedance can therefore be calculated as:

$$Z_c = \frac{hN}{2C} \quad (1.64)$$

Modelling of mechanical springs is performed in exactly the same way as the volume, since they also represent pure capacitances. The only difference is that it uses speed, instead of flow, and force, instead of pressure.

1.1.2 The transmission line as an integrator

In order to compare the transmission line with a true integration and the trapezoidal rule it is useful to draw the Bode- diagram of the Laplace transformed expressions. This is obtained by Laplace transforming (33). Using:

$$\mathcal{L}(x(nT+t)) = X e^{nsT} \quad (1.65)$$

The transmission line equation can be written:

$$P_1 e^{sT} = Z_c (Q_1 e^{-sT} + Q_1 e^{sT}) + P_1 e^{-sT} \quad (1.66)$$

Solving for P_1 and shifting the equation forward T yields:

$$P_1 = \frac{Q_1 Z_c (e^{2sT} + 1)}{e^{2sT} - 1} \quad (1.67)$$

The transfer function G is defined as:

$$G_{\text{line}} = \frac{P_1}{Q_1 Z_c} = \frac{1 + e^{-2sT}}{1 - e^{-2sT}} = \frac{e^{2sT} + 1}{e^{-2sT} - 1} \quad (1.68)$$

In the same way the transfer function for the trapezoidal rule is obtained as

$$G_{\text{trapez}} = \frac{Y}{F} = \frac{e^{hs} + 1}{2(e^{hs} - 1)} \quad (1.69)$$

The exact solution is of course

$$G_{\text{exact}} = \frac{1}{s} \quad (1.70)$$

Plotting the transfer functions (with $s = i\omega$) for trapez, the transmission line with $T = h$ and the exact solution yields up to $\omega = \frac{2\pi}{T}$.

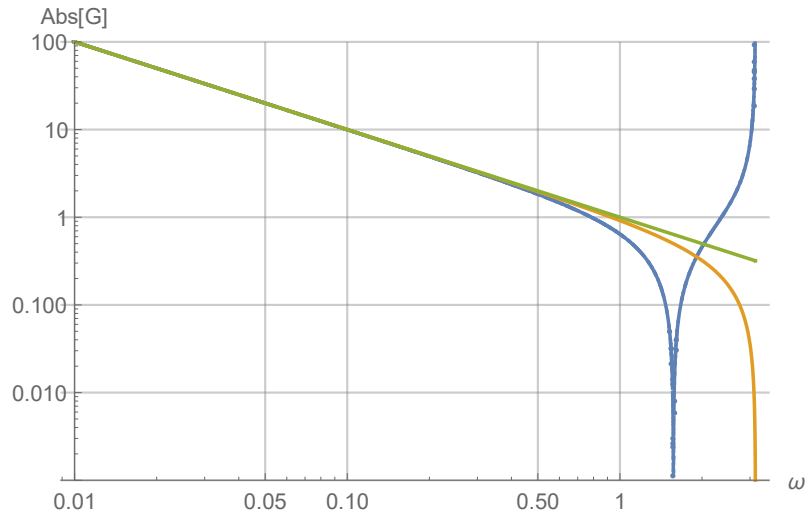


Figure 1.9

In the figure above the different transfer functions are shown. The frequency is non-dimensional such that it is normalized with respect to T . Since the transmission line transfer function is the same as that of the trapezoidal rule, but with half the frequency, the first resonant peak is visible in the spectrum. This resonance can give rise to oscillations during abrupt transients. Therefore, it is advantageous to introduce some damping in the transmission line in order to avoid high frequency oscillations at a frequency corresponding to $1/(2T)$ when both ends are closed. (In some way this oscillations can be regarded as heat, that in real systems can travel up into frequencies but here is captured at $1/(2T)$).

This can be implemented by introducing a slight filtering of the wave variables the characteristics [12]. Introducing the filtered wave variables as:

$$c_{f1}(t) = \alpha c_{f1}(t-h) + (1-\alpha)c_1(t) \quad (1.71)$$

$$c_{f2}(t) = \alpha c_{f2}(t-h) + (1-\alpha)c_2(t) \quad (1.72)$$

Care must, however, be taken not to introduce too much, since that would introduce dissipation in the system that might cause unwanted damping in the system behaviour, that would be an artefact from the solver.

The total set of equations for the transmission line element with filtering and Laplace transformed then becomes:

$$p_1(t) = c_{f1}(t) + Z_c q_1(t) \quad (1.73)$$

$$p_2(t) = c_{f2}(t) + Z_c q_2(t) \quad (1.74)$$

$$c_1(t) = p_2(t-h) + Z_c q_2(t-h) \quad (1.75)$$

$$c_2(t) = p_1(t-h) + Z_c q_1(t-h) \quad (1.76)$$

$$c_{f1}(t) = \alpha c_{f1}(t-h) + (1-\alpha)c_1(t) \quad (1.77)$$

$$c_{f2}(t) = \alpha c_{f2}(t-h) + (1-\alpha)c_2(t) \quad (1.78)$$

Laplace transformation yields:

$$P_1 = C_{f1} + Z_c Q_1 \quad (1.79)$$

$$P_2 = C_{2f} + Z_c Q_2 \quad (1.80)$$

$$C_1 = (P_2 + Z_c Q_2)e^{-sh} \quad (1.81)$$

$$C_2 = (P_1 + Z_c Q_1)e^{-sh} \quad (1.82)$$

$$C_{f1} = \alpha C_{f1} e^{-sh} + (1 - \alpha)C_1 \quad (1.83)$$

$$C_{f2} = \alpha C_{f2} e^{-sh} + (1 - \alpha)C_2 \quad (1.84)$$

With a blocked outlet ($Q_2 = 0$) and introducing the filter function $G_f(s)$ as:

$$C_{fi} = G_f(s)C_{fi} \quad (1.85)$$

Hence

$$G_f(s) = \frac{\alpha}{1 - (1 - \alpha)e^{-sh}} \quad (1.86)$$

This yields the set of equations of the transmission line as an integrator as:

$$P_1 = C_{f1} + Z_c Q_1 \quad (1.87)$$

$$P_2 = C_{2f} \quad (1.88)$$

$$C_1 = P_2 e^{-sh} \quad (1.89)$$

$$C_2 = (P_1 + Z_c Q_1)e^{-sh} \quad (1.90)$$

$$C_{f1} = G_f(s)C_1 \quad (1.91)$$

$$C_{f2} = G_f(s)C_2 \quad (1.92)$$

Solving for P_1 as a function of Q_1 yields:

$$G_{line} = \frac{P_1}{Z_c Q_1} = \frac{1 + G_f(s)e^{-2sh}}{1 - G_f(s)e^{-2sh}} \quad (1.93)$$

However, the filtering has some effect on the asymptotic behavior and in order to compensate for that a factor $1/(1 - \alpha)$ has to be introduced:

$$G_{line} = \frac{1}{(1 - \alpha)} \frac{1 + G_f(s)e^{-2sh}}{1 - G_f(s)e^{-2sh}} \quad (1.94)$$

The bode diagram G_{line} for this for $\alpha = 0$ and $\alpha = 0.2$ is plotted in Fig. 1.10 If the transmission line is to represent a capacitance e.g. if the transmission line is to represent a hydraulic volume or electric capacitance. Z_c should be selected using Eq. 1.34, and with $T = h$, as:

$$Z_c = h/C \quad (1.95)$$

However, with the correction factor for α it becomes

$$Z_c = \frac{h}{(1 - \alpha)C} \quad (1.96)$$

In addition there is also a parasitic unwanted inductance that becomes a modelling error. This is a function of time step

$$L_{parasitic} = Z_c h = \frac{h^2}{(1 - \alpha)C} \quad (1.97)$$

Note, that although this will introduce a numerical error, it will behave as a modelling error, which is more benign since conservation laws are upheld.

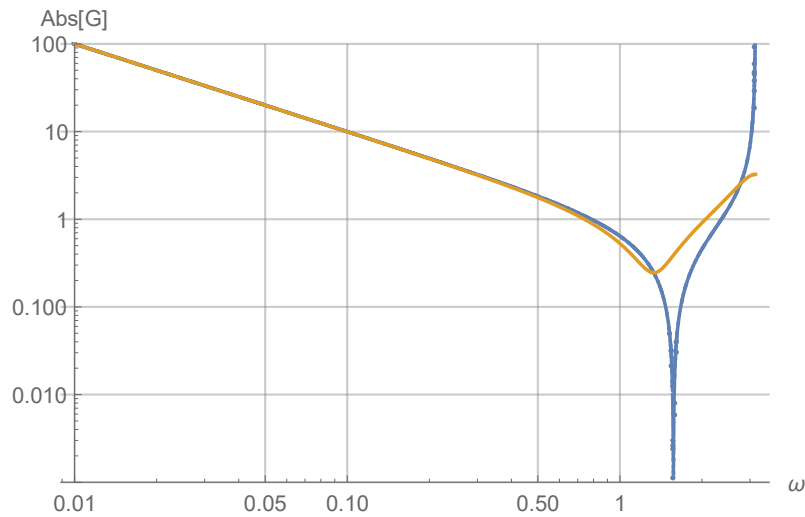


Figure 1.10: Bode diagram of the filtered transmission line element with $\alpha = 0$ and $\alpha = 0.2$.

1.1.3 Laminar Restrictor Connected to a Transmission Line

In this section we will look at the interaction between a transmission line and a laminar restrictor. This is an important special case since many components in some ways acts as restrictors e.g. valves, or at least have some part of their behavior described as resistive, e.g. the leakage in pumps, motors and pistons. It is therefore a case that can describe important aspects of the dynamic interaction between components and transmission lines.

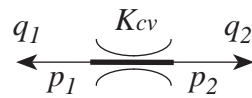


Figure 1.11: Laminar restrictor

The flow through the restrictor can be calculated as

$$q_2 = K_{cv}(p_1 - p_2) \quad (1.98)$$

K_{cv} is the flow-pressure coefficient of the restrictor. It is also possible to use the hydraulic resistance R instead. This is the inverse of the flow-pressure coefficient so that

$$R_v = 1/K_{cv} \quad (1.99)$$

In modelling hydraulic component the flow-pressure coefficient is usually preferred to the resistance since it can handle the extremes better. A closed valve for instance has $K_c = 0$. R on the other hand becomes infinite. We will, however, use R here since it has the same dimension as the characteristic impedance Z_c .

Equation (1.98) can then be rewritten as

$$q_2 = (p_1 - p_2)/R_v \quad (1.100)$$

When the restrictor is connected to transmission lines the following equations are solved for

the restrictor.

$$q_2 = (p_1 - p_2)/R_v \tag{1.101}$$

$$q_1 = -q_2 \tag{1.102}$$

$$p_1 = c_1 + Z_{c1}q_1 \tag{1.103}$$

$$p_2 = c_2 + Z_{c2}q_2 \tag{1.104}$$

The pressure difference can thus be calculated as

$$p_1 - p_2 = c_1 - c_2 - (Z_{c1} + Z_{c2})q_2 \tag{1.105}$$

The solution for q_2 is then

$$q_2 = \frac{c_1 - c_2}{R_v + Z_{c1} + Z_{c2}} \tag{1.106}$$

Note the similarity between equation (1.100) and (1.106). The only difference is that the pressures has been substituted with the characteristics and that the characteristic impedances has been added to the resistance.

The equations used in the component will thus be

$$q_2 = \frac{c_1 - c_2}{R_v + Z_{c1} + Z_{c2}} \tag{1.107}$$

$$q_1 = -q_2 \tag{1.108}$$

$$p_1 = c_1 + Z_{c1}q_1 \tag{1.109}$$

$$p_2 = c_2 + Z_{c2}q_2 \tag{1.110}$$

The corresponding block diagram can be seen in Fig. 1.12

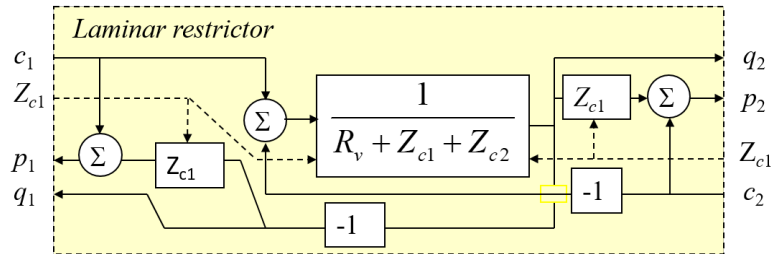


Figure 1.12: Block diagram of laminar restrictor.

The restrictor can then be connected to the line and a prescribed pressure as indicated in Fig 1.13 and the block diagram in Fig 1.14



Figure 1.13

In this way it is possible to assemble components and simulate the system. Note that there is a direct mapping between components and equations. This is a very important property for handling the complexity of large scale system simulation.

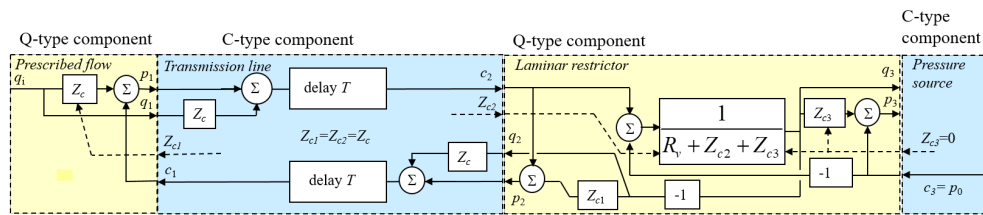


Figure 1.14: Transmission line connected to a laminar restrictor and a prescribed pressure component.

1.2 The Structure of the Simulation Model

Modelling of systems is essentially how to connect the different components so that they will interact in a proper way. The terminology used here, is related to the Hopsan software, although it is to a varying degree also applicable also to other simulation software.

1.2.1 The bi-directional delay graph

The bi-directional delay graph is a special case of symmetric directed graphs, see e.g. [2]. Symmetric directed graphs are network where connectors between nodes are bi-directional as in Fig. 1.15.

For simulation of systems the equations have to sorted for execution in each time step. This can lead to algebraic loops that cannot be resolved unless a time delay is introduced. This can lead to numerical instability if not implemented appropriately. Alternatively all equations from all system components are assembled into one system of equation that can then be solved. One way to avoid ad-hoc loop-breaking is to introduce time delays between some components. Here the components are divided into two types (C-type and Q-type) that are connected with bi-directional ports. When it comes to node connections there are two types of components. Those that are of a capacitive nature, where intensity variables are calculated, are called C-type components, and those that calculates flow variables such as flow and pressure or speed and force. These are called Q-type components.

The connections going from Q-type to C-type are associated with a time delay as in Fig. 1.16. This means that sorting for execution becomes trivial. For each step all C-type component are first executed (can be executed in parallel) and then all Q-type components (also possibly in parallel), Fig. 1.16. In this way all C-type and all Q-type components can be executed in parallel respectively. A modelling technique that allows for this is transmission line modelling (TLM), [1], [8], [13]. Here physically motivated time delays are introduced since wave-propagation is modelled in the C-components, between Q-components.

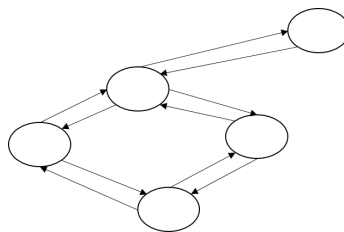


Figure 1.15: Symmetric directional graph.

During the simulation it is suitable to have all C-type components in one section and the Q-type components in another section. Since all communication between Q-type components will go through the C-type components there will be no communication between Q-type components in one time step. Consequently there are no restriction of the ordering of Q-type components, except for the case of a feedback controller. In a position servo the load position will be feed back to the servo

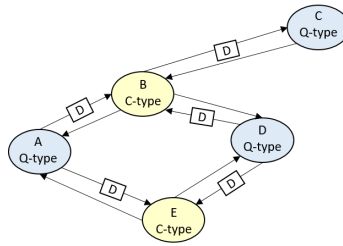


Figure 1.16: Bi-directional delay graph.

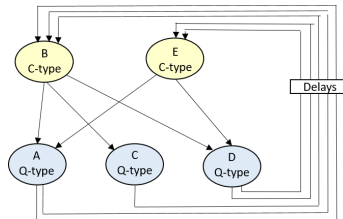


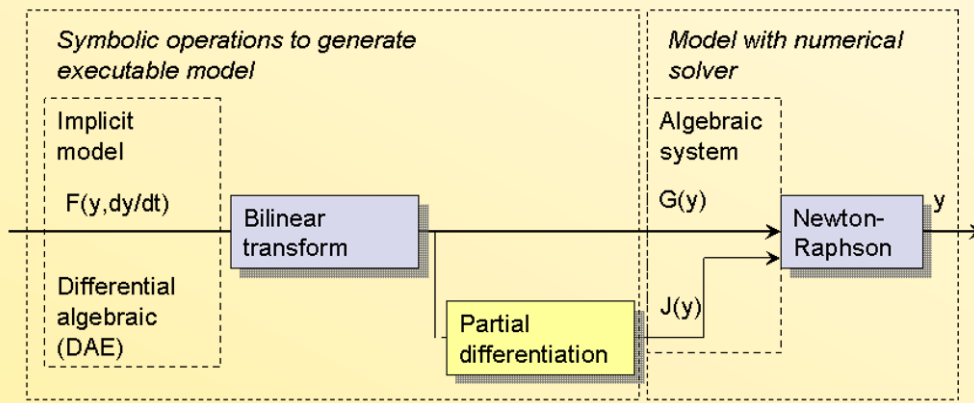
Figure 1.17: Rearranged Bi-directional delay graph. This can be executed explicitly every time step without iteration or algebraic manipulation

valve without any time delay. This aspect makes it recommendable to place the Q-type components that represents control elements such as valves last in the section of Q-type components.

When simulating systems the evaluation of the system is performed in two steps:

- The first step in the simulation loop is to calculate the variables from all C-type components (capacitive elements), such as volumes, springs and electric capacitances.
- The flows variables from the Q-type components can then be calculated. Step one is then repeated for the next time step.

Note, however, that a Q-type component may represent a system with an internal C-type (capacitive) component. This does not matter as long as all the external connecting nodes for the component are of the same type, ie calculates flows and pressure.



2. Differential Algebraic Equations

The most general representation of dynamic system is to represent it as a differential algebraic system. This means that the system is a combination of differential equations and algebraic equations.

The use of transmission line elements between components means that each component can be regarded as a separate unit that is numerically insulated from the rest of the system. It can therefore be solved independently from the rest of the system for each time step and can be modelled in any way that is convenient. This means that the components/sub systems can be models that are set up in a conventional way as conventional lumped parameter model. At each component between the lines the equation system 2.2 has to be solved.

$$\mathbf{f} = \begin{pmatrix} \mathbf{f}_a(x, \dot{\mathbf{x}}, \ddot{\mathbf{x}} \dots, \mathbf{q}, t) \\ \mathbf{f}_b(\mathbf{p}, \mathbf{c}, \mathbf{q}) \end{pmatrix} = 0 \quad (2.1)$$

Here $F_b(y)$ is the equations representing the boundary conditions. In general they can be written as

$$\mathbf{f}_b(\mathbf{q}) = \mathbf{p} - \mathbf{c} - \mathbf{Z}_c \mathbf{q} \quad (2.2)$$

where \mathbf{c} , \mathbf{p} , and \mathbf{q} are vectors containing the states of the adjacent nodes. \mathbf{Z}_c is a diagonal matrix containing the characteristic impedances. Introducing the state vector \mathbf{y} as a combination of \mathbf{x} and the boundary variables \mathbf{p} , \mathbf{c} and \mathbf{q} the equation system can be written as:

A general approach to represent a system is to represent it as a differential algebraic system. This also allows for algebraic loops. The simulation language Modelica is a language that is based on this form. In the general case the following system is considered.

$$\mathbf{f}(x, \dot{\mathbf{x}}, t) = 0 \quad (2.3)$$

However, (2.3) implies that the system essentially has to be written in state space form, something that may be considered as too limited. Many relationships are usually given in transfer

function form, which makes it more natural to allow for higher derivatives. The system can then instead be expressed as:

$$\mathbf{f}(y, \dot{y}, \ddot{y} \dots, t) = 0 \quad (2.4)$$

This also has the advantage that the variable vector is reduced, since y is shorter than x , y contains a subset of the states in x . It should, however, be pointed out that it is only possible to impose strong non-linearities (such as limitations on the state variables) represented in the y vector. Also all variables that are of any interest must be included in the y vector otherwise they will not be computed explicitly. Finally high order differentials should be avoided since the equations becomes numerically ill conditioned if the word length is limited. In order to solve the dynamic part of the system in a numerically stable way, the trapezoidal rule can be used. Using the trapezoidal rule the time differential is solved as

$$y(t+h) = y(t) + \frac{1}{2}h(\dot{y}(t) + \dot{y}(h+t)) \quad (2.5)$$

This is used in in-line integration when this is introduced as an additional equations for each state to be integrated. An alternative way is to use the trapezoidal rule is to reformulate it in the form known as the bilinear transform.

$$\frac{d}{dt} = \frac{2(1-q^{-1})}{h(1+q^{-1})} \quad (2.6)$$

where q in this context represents the time displacement operator such that:

$$qy = y(t+h) \quad (2.7)$$

Using the bilinear transform in (2.6) means that 2.4 can be rewritten as a function \mathbf{g} of $y(t)$ and old states $y(t-nh)$.

$$\mathbf{g}(y(t), y(t-h), \dots, y(t-nh), \mathbf{u}(t), \mathbf{u}(t-h), \dots, \mathbf{u}(t-nh), t) = 0 \quad (2.8)$$

When solving the system all the old values $y(t-h) \dots y(t-nh)$ can be regarded as constants since they have already been established in previous time steps. Likewise the input vector is also known. Equation 2.8 is therefore rewritten as:

$$\mathbf{G}(y(t), t) = 0 \quad (2.9)$$

In order to solve this system of equations in a numerically stable way, the Jacobian matrix is needed, which is defined as:

$$\mathbf{J}_{ijk} = \frac{\partial \mathbf{G}_i(y_k(t))}{\partial y_j} \quad (2.10)$$

The equation can then be solved numerically using Newton-Raphson iteration.

$$\mathbf{y}_{k+1} = \mathbf{y}_k(t) - \mathbf{J}_k(t)^{-1} \mathbf{G}(\mathbf{y}_k(t)) \quad (2.11)$$

Since an iterative procedure is used, there is a potential for performance loss due to the number of iterations needed to solve the system. However, the values from the previous time step can be used as start values.

$$\begin{aligned} \mathbf{y}_0(t) &= \mathbf{y}(t-h) \\ \mathbf{J}_0(t) &= \mathbf{J}(t-h) \end{aligned} \quad (2.12)$$

Equation (2.11) can then be used to solve this system in each time step. If the system is linear, the system can be solved in only one iteration, and it is usually sufficient with only one iteration even for non-linear systems, especially if a small time step is used. There are, however, situations when input signals changes suddenly, e.g. a valve is changed step wise during one time step, that requires more than one iteration. In practice, however, it has been found that two iterations increase the tolerance against non-linearities dramatically, while a further increase to three iteration gives only minor improvement. Two iterations have therefore been found to be something near to an optimum for almost all situations.

Although it is possible to use (2.11) directly it is wise to replace the inverse of the Jacobian by using LU-decomposition instead. There are also a few other actions that can be done in order to further enhance the efficiency of the solver. In general, however, the effort to solve the system, i.e. use solver that utilize sparsity in the Jacobian, since the evaluation time increases more than linear with the system size. However, using transmission line elements to partition a model, subsystems need not be to large.

Provided the system is reasonably linear (slow variation of J , (2.11)) is an $A(\alpha)$ -stable method [4]. However, in reality, rather large variations of J can be tolerated. Even pure discontinuities can also be handled satisfactory using the above approach, when fixed-time step is used (as in real-time simulation).

In a normal case the iteration quickly finds the right solution as in Fig. 2.1. Note tat for the one dimensional case Newton-Raphson can be written as:

$$y_{k+1} = y_k - \frac{g(y)}{g'(y)} \quad (2.13)$$

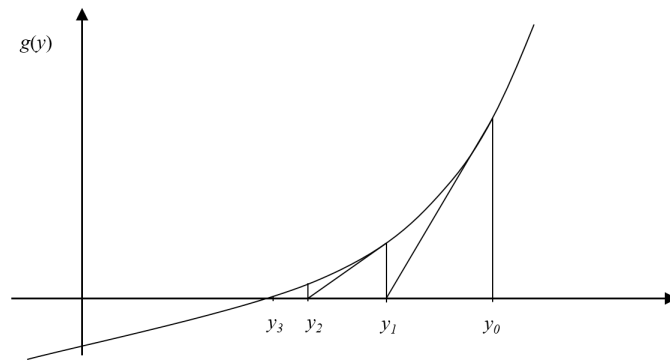


Figure 2.1: Convergence of Newton-Raphson iteration

There are, however, situations where even the Newton-Raphson method fails to converge. In Fig. 2.2 another scenario is shown. Here the non-linearity drives the method to diverge.

Figure 2.3 shows a scheme of the transformations involved to create an executable model. This was presented in [9]. The differential algebraic system DAE is transformed into time discrete form using bilinear transform. In Elmqvist et al [5] the concept of in-line integration is introduced, where extra equations are introduced to perform the integration. The scheme here is related but would be more appropriately called in-line transformation, and in-line integration can be viewed as a special case of that. The Jacobian J is obtained by symbolic partial differentiation of the time discrete system G . G and the Jacobian J are used to solve the system in each time step using the Newton-Raphson method for solving the system in each time step.

In order to deal with this overshoot it is possible to introduce the term α . If it is less than one the

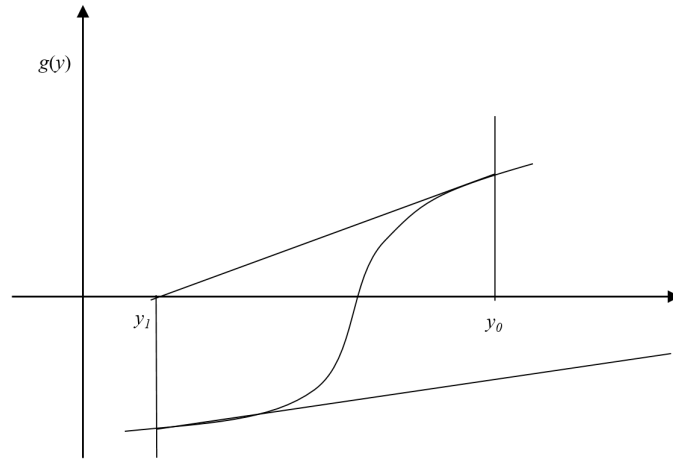


Figure 2.2: Newton-Raphson failing to converge.

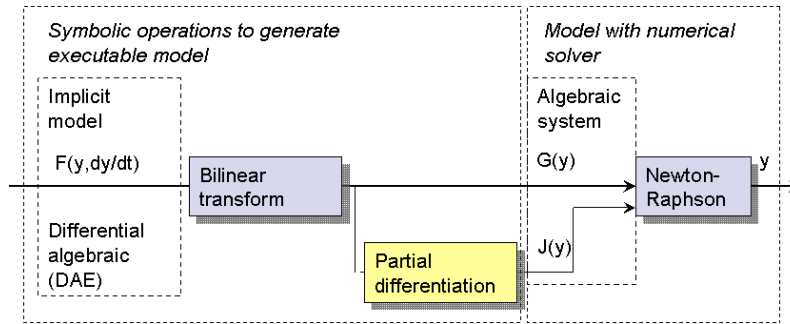


Figure 2.3: Scheme of transformations to generate a model including a solver using bilinear transformation and Newton-Raphson

convergence is slowed down but the chances of convergence increase.

$$\mathbf{y}_{k+1} = \mathbf{y}_k(t) - \alpha \mathbf{J}_k(t)^{-1} \mathbf{G}(\mathbf{y}_k(t)) \quad (2.14)$$

■ Example 2.1 Solving the laminar orifice as an algebraic system

In a previous section the simple laminar orifice was used as an example. Here it is shown how it would be solved using the more generic differential algebraic solver.

The equations are:

$$q_2 = (p_1 - p_2)/R_v \quad (2.15)$$

$$q_1 = -q_2 \quad (2.16)$$

$$p_1 = c_1 + Z_{c1}q_1 \quad (2.17)$$

$$p_2 = c_2 + Z_{c2}q_2 \quad (2.18)$$

Using the more general approach using Newton-Raphson. The system equations F and G would be (there is now dynamics):

$$\mathbf{G} = \mathbf{F} = \begin{pmatrix} p_1(-K_c) + p_2K_c + q_2 \\ q_1 + q_2 \\ -c_1 - q_1Z_{c1} + p_1 \\ -c_2 - q_2Z_{c2} + p_2 \end{pmatrix} \quad (2.19)$$

The equation of independent variables would be

$$\mathbf{y} = \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} \quad (2.20)$$

The Jacobian would then be

$$\mathbf{J} = \begin{pmatrix} 0 & 1 & -K_c & K_c \\ 1 & 1 & 0 & 0 \\ -Z_{c1} & 0 & 1 & 0 \\ 0 & Z_{c2} & 0 & 1 \end{pmatrix} \quad (2.21)$$

Using this approach would be marginally more computationally demanding than an analytical solution, since it is enough to use two iteration per time step. On the other hand it is using a completely general approach. ■

■ Example 2.2 Mechanical Inertia

This system of equations can be written as:

$$\mathbf{F} = \mathbf{0} \quad (2.22)$$

where

$$\mathbf{F} = \begin{pmatrix} M\dot{v}_2 + Bv_2 - (F_1 - F_2) \\ \dot{x}_2 - v_2 \\ v_1 - v_2 \\ x_1 + x_2 \\ F_1 - (c_{x,1} + v_1 Z_{cx,1}) \\ F_2 - (c_{x,2} + v_2 Z_{cx,2}) \end{pmatrix} \quad (2.23)$$

■

2.1 Event Free Modelling through functional programming

In real time simulation it is not practical to handle events by finding zero-crossings and restart solvers when e.g. limitations in variables are hit. Therefore other mechanisms needs to be used. Using the approach with differential algebraic equations that are solved through bilinear transformation and Newton-Raphson using an analytical Jacobian, it might seem that the Jacobian would be extremely difficult to derive, since most manually written models involves a great deal of conditions and jumps as they are written in a procedural style. This was shown in [10]. The introduction of an automated approach means, that a functional programming style is imposed (the function to represent the derivative of a function can always be defined although it may not always be an exact representation, e.g. in singular points). Therefore all conditions have to be represented by functions. The algorithm

```

if(cond)then
  a = a1
  b = b1
  c = c1
else
  a = a2
  b = b2
  c = c2
end if

```

can be transformed into

$$\begin{aligned} a &= \text{if}(\text{cond}, a_1, a_2) \\ b &= \text{if}(\text{cond}, b_1, b_2) \\ c &= \text{if}(\text{cond}, c_1, c_2) \end{aligned} \quad (2.24)$$

Since the If function is a piecewise continuous function the differential can be defined and can be used in the Jacobian. An advantage with this style of programming is that the variables a, b, and c in the example are forced to be defined for both cases, and cannot be forgotten as in normal procedural programming. This approach does not remove the events as such but there is no notion of events in the code since everything is hidden in the functions. The derivatives of the if function are:

$$\begin{aligned} \frac{\partial}{\partial y_1} \text{if}(\text{cond}, y_1, y_2) &= \text{if}(\text{cond}, 1, 0) \\ \frac{\partial}{\partial y_2} \text{if}(\text{cond}, y_1, y_2) &= \text{if}(\text{cond}, 0, 1) \end{aligned} \quad (2.25)$$

Another function that has proven very useful is the $\text{Limit}(x, x_{\min}, x_{\max})$ function.

$$\begin{aligned} \text{Limit}(x, x_{\min}, x_{\max}) &= x : x \in [x_{\min}, x_{\max}] \\ \text{Limit}(x, x_{\min}, x_{\max}) &= 0 : x \notin [x_{\min}, x_{\max}] \end{aligned} \quad (2.26)$$

The derivative of this function is defined as

$$\begin{aligned} \frac{d}{dx} \text{Limit}(x, x_{\min}, x_{\max}) &= 1 : x \in [x_{\min}, x_{\max}] \\ \frac{d}{dx} \text{Limit}(x, x_{\min}, x_{\max}) &= 0 : x \notin [x_{\min}, x_{\max}] \end{aligned} \quad (2.27)$$

Another function that can be useful is the "onPositive" function. For this function we have:

$$\begin{aligned} \text{onPositive}(x) &= 1 : x \in [0, \infty[\\ \text{onPositive}(x) &= 0 : x \notin [0, \infty[\\ \frac{d}{dx} \text{onPositive}(x) &= 0 \end{aligned} \quad (2.28)$$

The exact expression of the derivative of onPositive would have an infinite spike at precisely zero, this is not practical to use in any implementation, and this is therefore omitted.

Min and max functions

Discontinuities can pose numerical problems (although not always) but if possible it is advisable to replace them with a continuous functions that blend the two functions together. In many cases the real function does not even have a sharp discontinuity and there should be a smooth blending between the two functions. If a function is defined in one region by a function $f_a(x)$ and in another by $f_b(x)$ and the function is the largest of these values. I.e.

$$f(x) = \max(f_a(x), f_b(x)) \quad (2.29)$$

we can use:

$$f(x) = (f_a^\kappa + f_b^\kappa)^{1/\kappa} \quad (2.30)$$

With a high value of κ the largest value quickly becomes very dominant. If

$$f_a(x) = x \quad (2.31)$$

and

$$f_b(x) = 1 \quad (2.32)$$

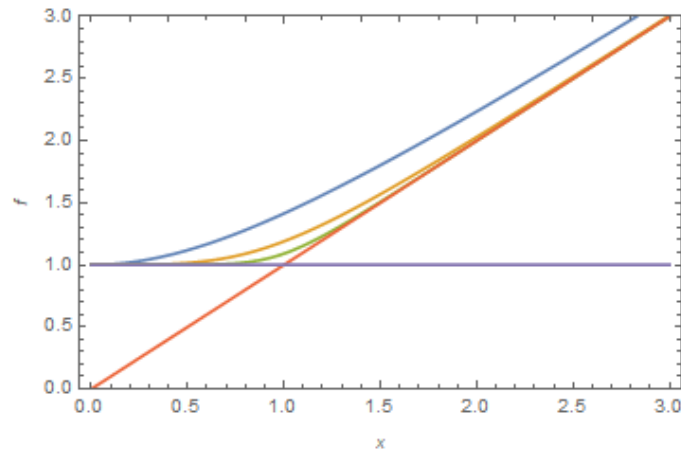


Figure 2.4: Function that takes the maximum of two functions with a smooth transition, for $\kappa = 2$, $\kappa = 4$ and $\kappa = 6$.

The corresponding plots is shown in Fig. 2.4: As can be seen from the plots a typical value for κ could be 4. If a function closer to the exact max-function is wanted a higher value it is just a matter of increasing κ .

In a similar way an approximation to the a min-function can be defined.

$$f(x) = \min(f_a(x), f_b(x)) \quad (2.33)$$

Here the problem is first made into the max-function by inverting the functions and than inverting again after the summation. 2.33.

$$f(x) = \frac{1}{\left(\frac{1}{f_a(x)^\kappa} + \frac{1}{f_b(x)^\kappa}\right)^{1/\kappa}} \quad (2.34)$$

where n is an exponent that for a very large value makes the transition sharp. To avoid singularities if any of the functions are zero it can be rewritten as:

$$f(x) = \frac{f_a(x)f_b(x)}{(f_a(x) + f_b(x)^\kappa)^{1/\kappa}} \quad (2.35)$$

The mixed function would then be:

$$f(x) = \frac{1}{\left(\frac{1}{x} + 1\right)^{1/\kappa}} \quad (2.36)$$

The function is plotted in Fig. 2.5 for the values of $\kappa = 2, 4$, and 6 along with f_a and f_b .

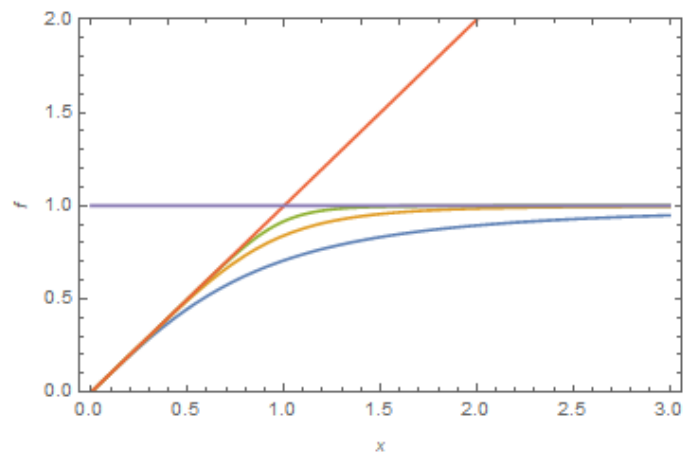


Figure 2.5: Function that takes the minimum of two functions with a smooth transition, for $\kappa = 2$, $\kappa = 4$ and $\kappa = 6$.

The logistic function

A general way of blending two functions is to use the logistic function. This can be written as

$$f_{\text{logistic}}(x) = \frac{1}{1 + e^{\frac{2(x-x_0)}{x_w}}} \quad (2.37)$$

The result for $x_0 = 0$ and $x_w = 1$ is shown in Fig. 2.6.

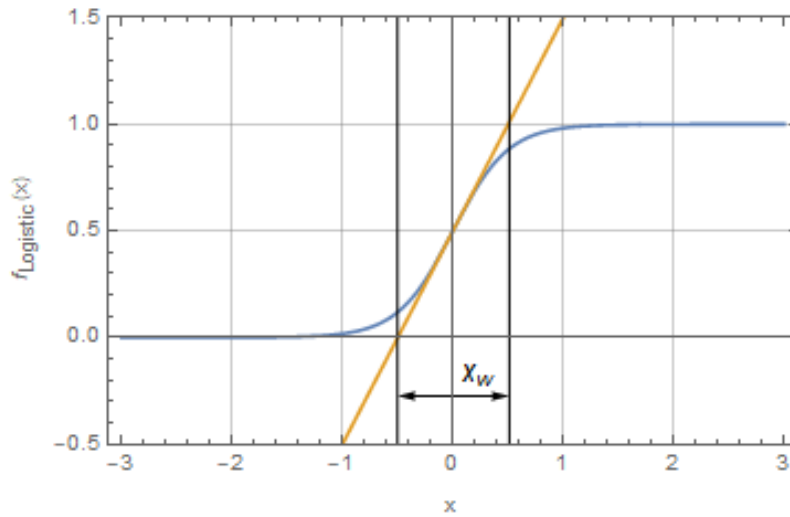


Figure 2.6: The logistic function

To blend two functions at an arbitrary point we can use the logistic function in the following way:

$$f(x) = f_L(x)f_a(x) + (1 - f_L(x))f_b(x) \quad (2.38)$$

Using the same functions as before and with $x_w = 0.1$ and $x_0 = 1.5$ the result in Fig 2.7. is obtained.

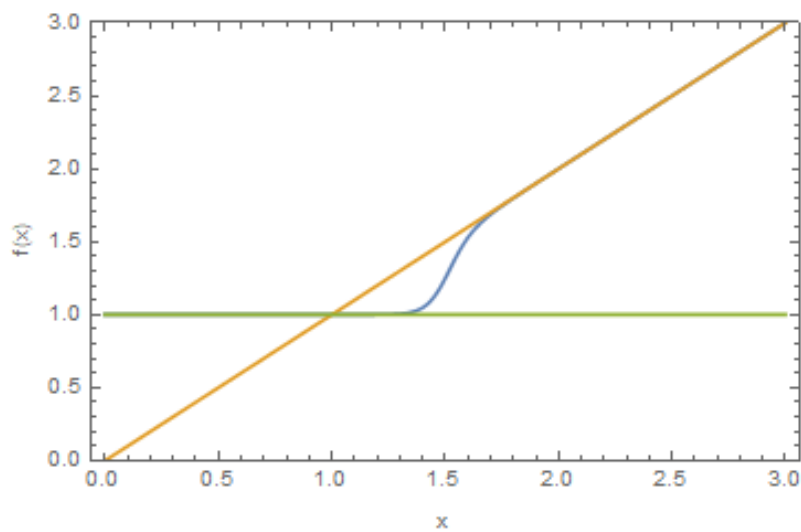


Figure 2.7

The signed square root function

A function that is very useful in particular in fluid power systems is the SigSqr function. This is useful to represent flow in sharp edged orifices.

$$\text{SigSqr}x = (\text{sign}(x))\sqrt{|x|} \quad (2.39)$$

Its derivative is

$$\text{SigSqr}'(x) = \frac{1}{2\sqrt{|x|}} \quad (2.40)$$

In order to avoid division with zero a small linear region is introduced. This can also be physically motivated as there is always a small linear region even in sharp orifices. It is therefore denoted SigsqrL.

$$\text{SigSqrL}(x, x_0) = \text{sign}(x) \left(\frac{|x|^\kappa}{|x|^{\kappa/2} + x_0^\kappa} \right)^{1/\kappa} \quad (2.41)$$

This function is shown in Fig. for the values $x_0 = 0.3$ and $\kappa = 4$

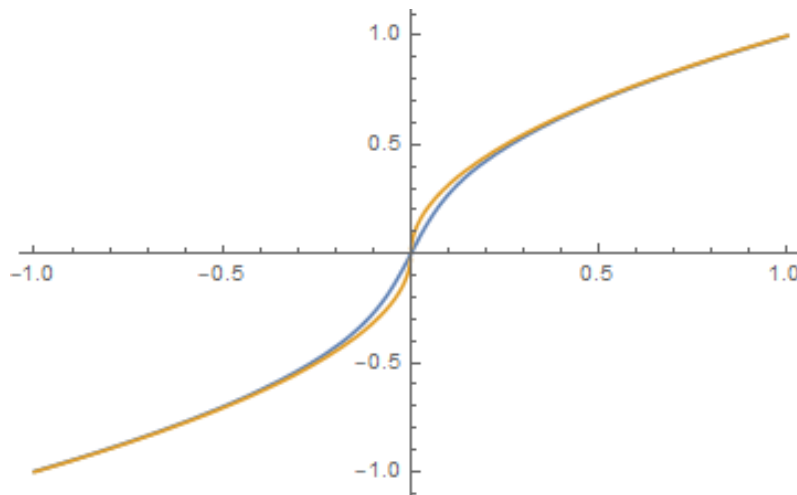


Figure 2.8: The modified signed square root function that can be used to represent the flow in an orifice with sharp edges.

■ Example 2.3 Hydraulic Pressure Relief Valve

A very common element in hydraulic systems is a pressure relief valve. It is a rather simple component, yet it contains several difficulties from a modelling point of view. It includes a sharp edged orifice and also discontinuities in the form of limitations on the stroke of the valve spool.

The equation describing this component are:

$$\mathbf{F} = \begin{pmatrix} m_v \text{der}(v_v) + b_v v_v + k_e x_v - A_v (p_s - p_t - p_{ref}) \\ m_v \text{der}(\text{der}(x_v)) + b_v \text{der}(x_v) + k_e x_v - A_v (p_s - p_t - p_{ref}) \\ C_d w x_v \text{SigSqr}\left(\frac{2}{\rho}(p_s - p_t)\right) - q_t \\ q_s - q_t \\ p_s - (c_s + z_{cs} q_s) \\ p_t - (c_t + z_{ct} q_t) \end{pmatrix} \quad (2.42)$$

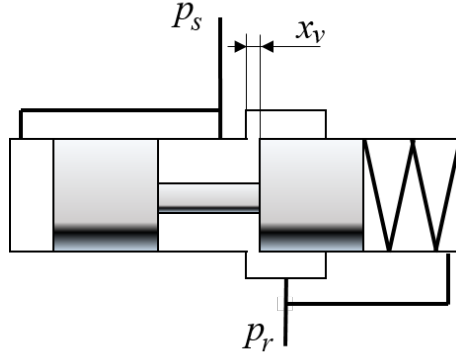


Figure 2.9: Pressure relief valve.

Here

$$w = 2\pi d_v \quad (2.43)$$

$der()$ is the time differential operator. Bilinear transform yields:

$$\mathbf{G} = \begin{pmatrix} -A_v h^2 p_1 + A_v h^2 p_2 + A_v h^2 p_{ref} + (4M_v + 2B_v h + h^2 k_e) x_v + \\ \quad + DS(-2A_v h^2 p_1 + 2A_v h^2 p_2 + 2A_v h^2 p_{ref} + 2h^2 k_e - 8M_v), 1) + \\ \quad + DS(-A_v h^2 p_1 + A_v h^2 p_2 + A_v h^2 p_{ref} + \\ \quad + 4M_v x_v - 2B_v h x_v + h^2 k_e x_v, 2) \\ B_v v_v h + 2v_v M_v - A_v h p_1 + A_v h p_2 + A_v h p_{ref} + h k_e x_v + \\ \quad + DS(B_v v_v h - 2v_v M_v - A_v h p_1 + A_v h p_2 + A_v h p_{ref} + h k_e x_v, 1) \\ C_q w x_v \text{SigSqr}(\frac{2}{\rho}(p_s - p_t)) - q_t \\ q_s - q_t \\ p_s - (c_s + z_{cs} q_s) \\ p_t - (c_t + z_{ct} q_t) \end{pmatrix} \quad (2.44)$$

The vector of independent variables for the system of equations is:

$$y = (x_v, v_v, q_t, q_s, p_s, p_t)^T \quad (2.45)$$

There are limits on x_v that also influence x_v . These limitations are introduced after the bilinear transformation of (26). It means limiting the value of and to make sure the speed is set to zero when limitations are active. This also explains why the equation of motion is written two times instead of using the speed to integrate the position. Such a scheme would get the position stuck.

$$\mathbf{G} = \begin{pmatrix} x_v - \text{Limit}(\frac{1}{(4M_v + 2B_v h + h^2 k_e)} (-A_v h^2 p_1 + A_v h^2 p_2 + A_v h^2 p_{ref} + \\ \quad + DS(-2A_v h^2 p_1 + 2A_v h^2 p_2 + 2A_v h^2 p_{ref} + 2h^2 k_e - 8M_v), 1) + \\ \quad + DS(-A_v h^2 p_1 + A_v h^2 p_2 + A_v h^2 p_{ref} + 4M_v x_v - 2B_v h x_v + h^2 k_e x_v, 2), 0, x_{v,\max}) \\ v_v - \frac{d}{dx} \text{Limit}(-A_v h p_1 + A_v h p_2 + A_v h p_{ref} + h k_e x_v + DS(B_v v_v h - 2v_v M_v - A_v h p_1 + A_v h p_2 + A_v h p_{ref} + h k_e x_v, 1) \\ C_q w x_v \text{SigSqr}(\frac{2}{\rho}(p_s - p_t)) - q_t \\ q_s - q_t \\ p_s - (c_s + z_{cs} q_s) \\ p_t - (c_t + z_{ct} q_t) \end{pmatrix} \quad (2.46)$$

The Hopsan simulation package is used to simulate the pressure relief valve is simulated in a simple system where it limits the pressure in a volume.

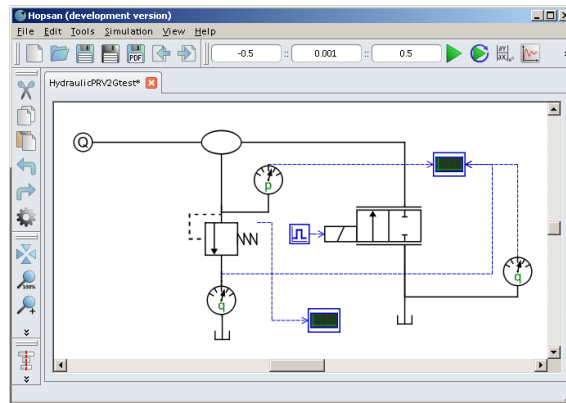


Figure 2.10

For the simulation the right valve is suddenly opened after 0.01 seconds, and then closed after 0.3 seconds. The time step for the simulation was 0.001 seconds. The simulation results shows that the pressure relief valve reach the extremes during the transient, and handle that perfectly using a fixed time step.

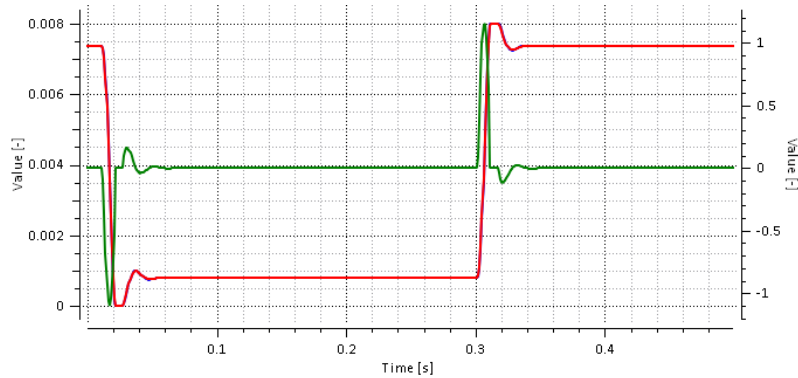


Figure 2.11: Spool position as a function of time. It hits limitation in both ends of the stroke.

■

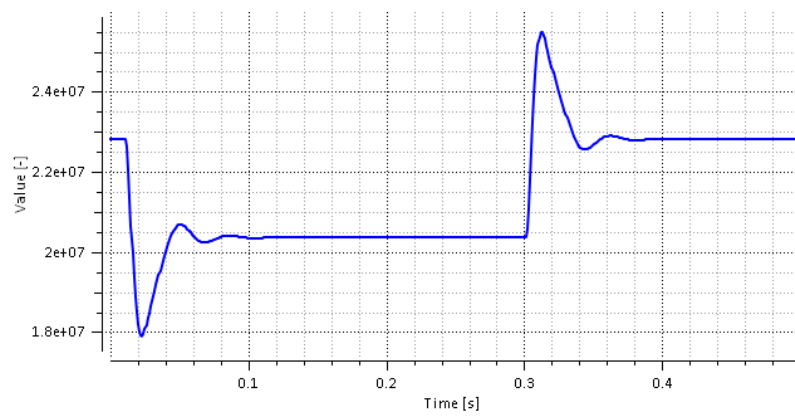
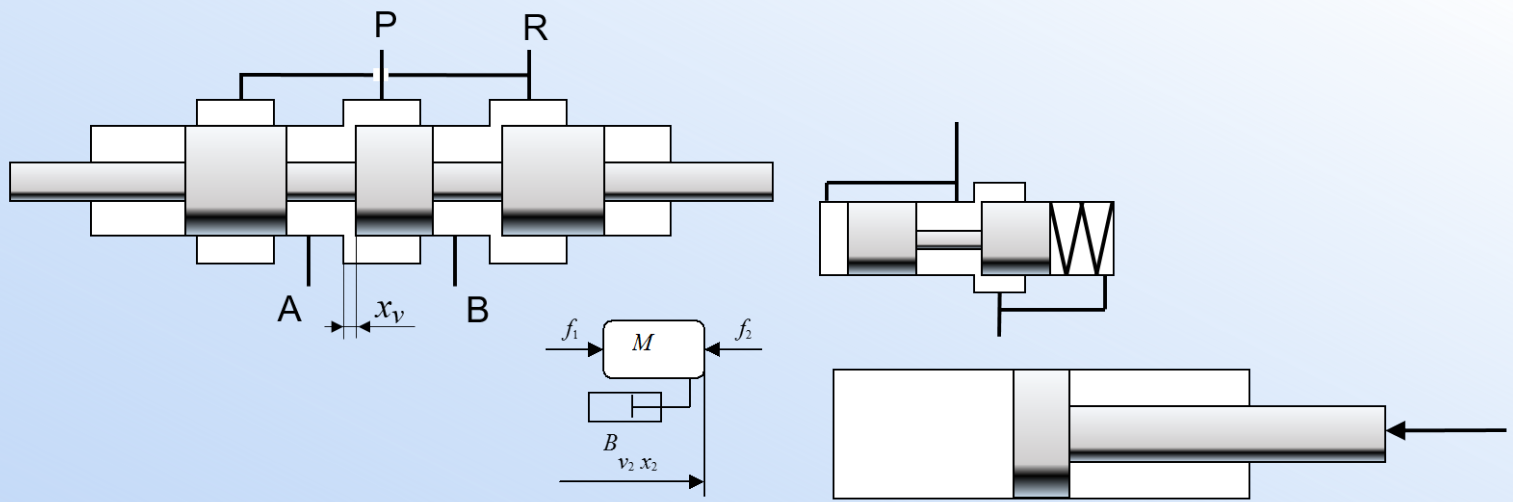


Figure 2.12: Pressure as a function of time.



3. Modelling of Hydraulic components

P. KRUS ¹

3.1 Introduction

Modelling of components generally means that behaviour at a detail equation level need to be established, in contrast to system modelling that is the assembly of predefined components/subsystems into systems. This means that component modelling requires a different skill set. However, there is a grey zone since some components can be build out of more primitive components. E.g. a simple equation based pump model is rather straightforward to established, while a more detailed model modelling flow pulsation can be made up as a system of pistons and valves. This also point to the question of what is an appropriate level of detail in a simulation model. It thus becomes very important to realize the purpose of the simulation model. In conceptual design of systems, rather simple, low-fidelity (LF) models are needed. These are models that can represent the basic intended functions of a component, as well as being able to represent the most important secondary effect, e.g. basic losses. The reason to have simple models is not just a matter of using less resources in defining them. At the conceptual stage, geometry and other details, of the component in the system may not be known. Eg, it is known that a pump is needed but not what type, and certainly not the exact type. Hence the extra level of parameters that needs to be defined, would not be known at this stage, so a detailed model would just introduce more uncertainty. Furthermore, at the conceptual stage simulation based optimization is a powerful tool, and to be efficient, simple models that can be run thousands of times in a manageable amount of time are needed.

Looking at the design of an individual component a much higher level of detail is of course needed, but it can be studied in a system where the boundary conditions are made by low-fidelity models. For component design medium-fidelity models or even high-fidelity (HF) models including field equation (CFD, FEM) may be used, alternatively high fidelity models can be used to establish parameters in a medium fidelity model. In this chapter, modelling of low-fidelity models is

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discussed.

Definitions of fidelity levels

A working definition for fidelity models used here is:

- Low fidelity models are useful in conceptual design when details of components are not know.
- Medium fidelity models are useful in detail design when geometry of components are known or a matter of design.
- High fidelity models can be useful in component design,

3.2 The Laminar Restrictor

In order to demonstrate the principle of component modelling the very simple laminar orifice is used as an example. The pressure-flow relationship is:

$$q_2 = (p_1 - p_2)/R_v \quad (3.1)$$

The following equations are solved at the component (connected to lines)

$$q_2 = (p_1 - p_2)/R_v \quad (3.2)$$

$$q_1 = -q_2 \quad (3.3)$$

$$p_1 = c_1 + Z_{c1}q_1 \quad (3.4)$$

$$p_2 = c_2 + Z_{c2}q_2 \quad (3.5)$$

Here R_v is the resistance of the orifice, Z_{c1} and Z_{c2} are the characteristic impedances of the lines connected to the orifice.

Being a non-dynamic linear system, these equations can be solved for p_1 and p_2 , The equations used in the subroutines will thus be

$$q_2 = \frac{c_1 - c_2}{R_v + Z_{c1} + Z_{c2}} \quad (3.6)$$

$$q_1 = -q_2 \quad (3.7)$$

$$p_1 = c_1 + Z_{c1}q_1 \quad (3.8)$$

$$p_2 = c_2 + Z_{c2}q_2 \quad (3.9)$$

This is a Q -component since flows, q_1 and q_2 , are outputs and the wave variables, c_1 and c_2 , are inputs. A comparison with eq (3.6) and eq (3.2) shows that the adaptation to transmission lines has the same effect on the equations as adding restrictors with the resistance Z_c .

3.2.1 Bond-graphs in Transmission Line Modelling

Bond-graph modelling is a widely used technique for describing systems both at a system level and at a detailed level. It is therefore of some interest to investigate how the bond-graph method relates to distributed modelling. As an example the laminar restrictor is used.

From Eq (3.6) it is evident that the characteristic impedances of the lines can be treated as resistances in series with the component. With is approach it is straightforward to use Bond-graphs to describe components/subsystems on a detailed equation level. The Bond-graph description of the orifice is shown in Fig 3.1. This Bond-graph is ill formulated and has to be rearranged if it should be possible for the causalities to be defined in a consistent way. The modified Bond-graph with the causality bars is shown in Fig 3.2. In many valves, orifices with nonlinear pressure-flow relationships are present. Therefore, in complex components, with several restrictors in series and in parallel, it often becomes impossible to solve the equations analytically.

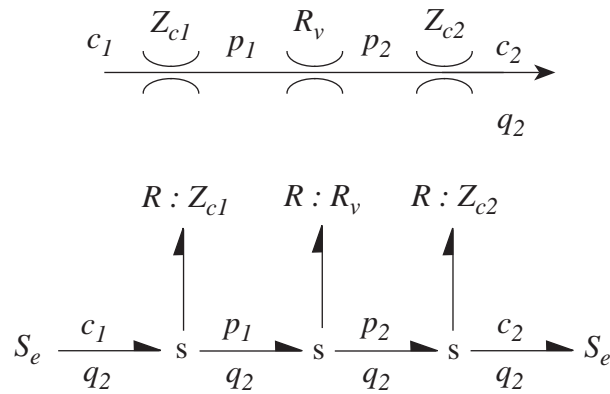


Figure 3.1: Bond-graph of restrictor including the adjacent characteristic impedances.

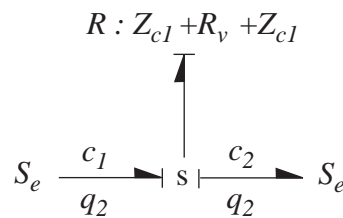


Figure 3.2: Reformulated Bond-graph of restrictor with causality bars

There is, however, an alternative way to handle this situation. This is to never assign the causalities. The Bond-graph is then used only to describe the relationships, without actually solving the equations (which has to be done in order to assign the causality bars in a consistent way). Instead, the (sub)system is represented as a differential algebraic system where no causality is represented. In the transmission line, however, causality is very much present but are completely determined by the underlying physics of the line. The reason for this, is that it is only in the transmission line that the spatial dimension is represented, and it is only this fact that introduce causality explicitly.

3.3 Turbulent restrictor

A very important component is the turbulent restrictor.



Figure 3.3: Turbulent orifice

The flow in a turbulent restrictor with a positive pressure drop can be expressed as:

$$q_2 = C_q A_0 \sqrt{\frac{2}{\rho} (p_1 - p_2)} \tag{3.10}$$

Here C_q is the discharge coefficient, A_0 is the crossection area and ρ is the density of the fluid. This

can also be written as:

$$q_2 = K_s \sqrt{p_1 - p_2} \quad (3.11)$$

Where K_s is the turbulent flow coefficient. This component is of the Q-type and should be connected to C-type components such as volumes and lines or nothing. The following equations therefore need to be solved

$$q_2 = \begin{cases} K_s \sqrt{p_1 - p_2} & \text{if } p_1 > p_2 \\ -K_s \sqrt{p_2 - p_1} & \text{if } p_1 < p_2 \end{cases} \quad (3.12)$$

$$q_1 = -q_2 \quad (3.13)$$

$$p_1 = c_1 + Z_{c1} q_1 \quad (3.14)$$

$$p_2 = c_2 + Z_{c2} q_2 \quad (3.15)$$

Equation (3.14) and (3.15) in (3.12) and with

$$Z_{ct} = Z_{c1} + Z_{c2} \quad (3.16)$$

yields

$$q_2 = \begin{cases} K_s \sqrt{c_1 - c_2 - Z_{ct} q_2} & \text{if } p_1 > p_2 \\ -K_s \sqrt{c_2 - c_1 + Z_{ct} q_2} & \text{if } p_1 < p_2 \end{cases} \quad (3.17)$$

The square of these are

$$q_2^2 = \begin{cases} -K_s^2 Z_{ct} q_2 + K_s^2 (c_1 - c_2) & \text{if } p_1 > p_2 \\ K_s^2 Z_{ct} q_2 - K_s^2 (c_1 - c_2) & \text{if } p_1 < p_2 \end{cases} \quad (3.18)$$

Who can be solved for q_2 . Note that $c_1 - c_2$ always has the same sign as $p_1 - p_2$ since $q_2 = 0$ when $p_1 - p_2 = 0$. This means that the characteristics can be used to determine the direction of flow. This finally yields that the equations for the orifice can be solved analytically.

$$q_2 = \begin{cases} \frac{K_s^2 Z_{ct}}{2} (\sqrt{1 + \frac{4(c_1 - c_2)}{K_s^2 Z_{ct}^2}} - 1) & \text{if } c_1 > c_2 \\ \frac{K_s^2 Z_{ct}}{2} (1 - \sqrt{1 + \frac{4(c_1 - c_2)}{K_s^2 Z_{ct}^2}}) & \text{if } c_1 < c_2 \end{cases} \quad (3.19)$$

$$q_1 = -q_2 \quad (3.20)$$

$$p_1 = c_1 + Z_{c1} q_1 \quad (3.21)$$

$$p_2 = c_2 + Z_{c2} q_2 \quad (3.22)$$

3.4 Geometry Based Model of an Orifice

In the previous examples the orifice was just represented with coefficients. An alternative approach is to base the model on the geometry of the orifice. In Fig. 3.4 a orifice with geometric parameters is shown. This can be considered as a medium fidelity model, since it requires a known geometry. This model is based on a combination of equations for different regimes. These can be found in the classic text book by Merrit [14]. These are:

Laminar flow through a slit conduit:

$$q_a = \frac{d_h^2 A_0}{48 l_0 \eta} (p_1 - p_2) \quad (3.23)$$

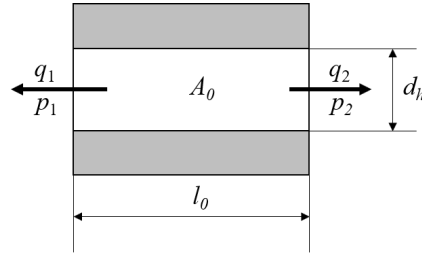


Figure 3.4: Parameters for a geometry based orifice model.

Here A_0 is the cross section area, l_0 is the length of the conduit, η is the dynamic viscosity, d_h is the hydraulic diameter:

$$d_h = 4A_0/S_0 \quad (3.24)$$

Here S_0 is the perimeter of the hole. Note that for a circular hole d_h is the actual diameter. Laminar flow through a circular hole can be calculated as:

$$q_a = \frac{d_h^2 A_0}{32l_0\eta} (p_1 - p_2) \quad (3.25)$$

These two can be combined to form a general expression.

$$q_a = \frac{d_h^2 A_0}{(48 - 16s_f)l_0\eta} (p_1 - p_2) \quad (3.26)$$

Here s_f is a correction factor that is equal to one for the circular hole and zero for a slit. Note that this expression is not valid for very small values of l_0 . For a very short orifice the following expression can be used.

$$q_b = \frac{\delta^2 d_h A_0}{\eta} (p_1 - p_2) \quad (3.27)$$

Here δ is the laminar flow coefficient. A typical value is $\delta = 0.157$

For high Reynolds numbers the flow becomes turbulent. The turbulent flow through an orifice is:

$$q_c = A_0 C_d \sqrt{\frac{2(p_1 - p_2)}{\rho}} \quad (3.28)$$

The resulting flow would be the minimum of all these flows. A simple way to combine these expressions is to create a function that takes a smoothed minimum value from these three expressions. This can be expressed as:

$$q_2 = \frac{1}{\left(\left(\frac{1}{q_a} \right)^\kappa + \left(\frac{1}{q_b} \right)^\kappa + \left(\frac{1}{q_c} \right)^\kappa \right)^{1/\kappa}} \quad (3.29)$$

Inserting 3.26, 3.27 and 3.28 into 3.29 and rearranging to avoid division with zero at zero pressure drop.

$$q_2 = \frac{p_1 - p_2}{\left(\left(\frac{(48 - 16s_f)l_0}{d_h^2 A_0} \right)^\kappa + \left(\frac{\eta}{2\delta^2 d_h A_0} \right)^\kappa + \left(\frac{\rho |p_1 - p_2|}{2(A_0 C_d)^2} \right)^{\kappa/2} \right)^{1/\kappa}} \quad (3.30)$$

Fig 3.5 shows the different component of the flow equations for the following values:

$$\begin{aligned} d_h &= 0.0025\text{m} \\ A_0 &= 1 \times 10^{-6}\text{m}^2 \\ \delta &= 0.157 \\ \eta &= 0.12\text{Ns/m}^2 \\ l_0 &= 0.5 \times 10^{-3}\text{m} \\ \rho &= 855\text{kg/m}^3 \\ \kappa &= 4 \end{aligned}$$

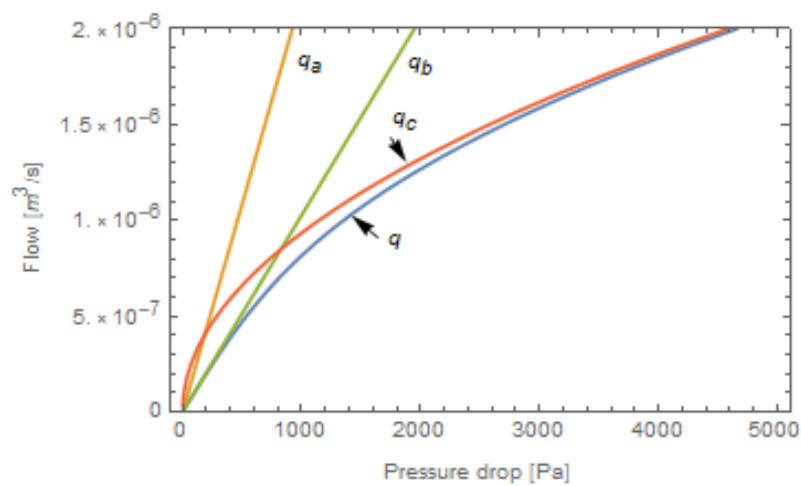


Figure 3.5: The different components of flow in the geometry based model of a restrictor.

3.5 Modelling of a hydraulic cylinder

There is a transformation of power in the cylinder from fluid to mechanical which means that the characteristics and impedances has to be converted.

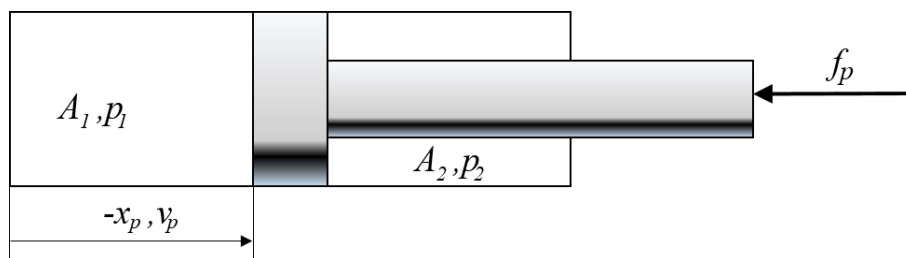


Figure 3.6: Hydraulic cylinder

$$\begin{aligned} f_p &= A_1 p_1 - A_2 p_2 \\ q_1 &= A_1 v_p \\ q_2 &= -A_2 v_p \end{aligned} \tag{3.31}$$

where v_p is the piston speed (\dot{x}_p). Introducing

$$f_p = c_x + z_{cx}v_p \quad (3.32)$$

(3.32) and (1.53) in (3.31) yields

$$\begin{aligned} c_x + z_{cx}v_p &= A_1(c_1 + Z_{c1}q_1) - A_2(c_2 + Z_{c2}q_2) \\ c_x + z_{cx}v_p &= A_1(c_1 + Z_{c1}A_1v_p) - A_2(c_2 - Z_{c2}A_2v_p) \end{aligned} \quad (3.33)$$

identification yields

$$\begin{aligned} c_x &= A_1c_1 - A_2c_2 \\ Z_{cx} &= A_1^2Z_{c1} + A_2^2Z_{c2} \end{aligned} \quad (3.34)$$

Here the flows, q_1 and q_2 and the speed v_p (and position x_p) are inputs and the fluid wave variables c_1 and c_2 and the mechanical wave variable c_x are outputs. This means that this is a *C-component*. Limitations of the stroke are easiest to implement in the Q-component representing the mechanical load.

The characteristic impedance for each volume is a function of time step and the capacitance of the volume.

$$Z_{ci} = h/C_i \quad (3.35)$$

The capacitance is a function of volume V_i of each volume, and the effective bulk modulus β_e .

$$C_i = V_i/\beta_e \quad (3.36)$$

The volume depends on the stroke x_p and the cross section area A_p .

$$V_i = (x_{0i} - x_p)A_p \quad (3.37)$$

Here x_{0i} represents the dead volume of the cylinder. It could also represent volumes in lines between valves and cylinder. Note that the sign of the piston position is negative. This is because speed is an input and hence should have a positive sign going into the cylinder. Consequently the position also gets a negative sign then. Hence $x_p = [-x_{p,max}, 0]$. Note that no friction has been mentioned here. Because of the causality this has to be modelled on the load attached to the piston.

3.6 Modelling of Mechanical Subsystems

An important aspect of simulation of mobile hydraulics is that the mechanical structure and its connections to the hydraulic actuators must be described. This can also be accommodated within the framework of TLM. A spring can be modelled exactly as a short pipeline but with force instead of pressure and speed instead of flow.

An inertia load with damping can be described so that it can be connected to transmission line elements, if the following equations are used.

$$M\dot{v}_2 = -Bv_2 + f_1 - f_2 \quad (3.38)$$

$$\dot{x}_2 = v_2 \quad (3.39)$$

$$v_1 = -v_2 \quad (3.40)$$

$$x_1 = -x_2 \quad (3.41)$$

$$f_1 = c_{x,1} + v_1Z_{cx,1} \quad (3.42)$$

$$f_2 = c_{x,2} + v_2Z_{cx,2} \quad (3.43)$$

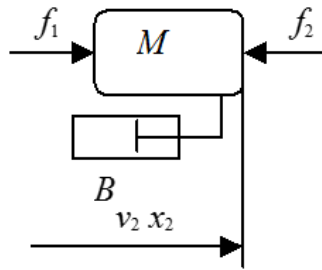


Figure 3.7: Mechanical inertia load

In this case the equations can also be solved analytically to obtain the following set of independent equations:

$$\dot{v}_2 = \frac{-v_2 (Z_{cx,1} + Z_{cx,2} + B) + c_{x,1} - c_{x,2}}{M} \tag{3.44}$$

$$\dot{x}_2 = v_2 \tag{3.45}$$

$$v_1 = -v_2 \tag{3.46}$$

$$x_1 = -x_2 \tag{3.47}$$

$$f_1 = c_{x,1} + v_1 Z_{cx,1} \tag{3.48}$$

$$f_2 = c_{x,2} + v_2 Z_{cx,2} \tag{3.49}$$

These equations can be solved using bilinear transform which corresponds to the implicit trapezoidal rule.

An important observation from equation 3.49 is that B and Z_{cx} have the same influence. It is a general observation that Z_c has the same effect as damping in components. The reason for this is that the phase shift caused by the fact that c represents old information is compensated for by the additional damping, and this is a good way to check if a model seems right. The pressure/force is replaced by a characteristic and damping is added to the model.

When it comes to sign conventions they are the same as for flows. That is, speeds entering a spring are positive and speeds leaving an inertia load are also positive. This may cause some confusion when it comes to positions who are the integrals of speeds. The convention is that positions on two sides of an inertia are mirror images of each other. This is important to realize when assigning start values to the loads. They should have the same value but different sign. Furthermore, this implicates that the position at the piston will always have negative sign. In Fig 3.8 a block diagram of the (Laplace transformed) equations of an inertia load connected to a mechanical spring is shown.

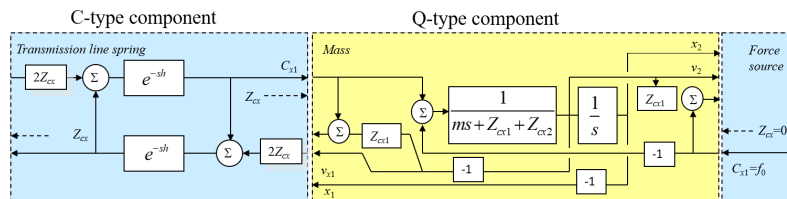


Figure 3.8: Block diagram showing a mass connected to a TLM spring and a force source. (Dealing with limitations, numerical drift etc are not included).

3.6.1 Dealing with limitations

Sometimes a variable is subjected to limitations. In the case of the inertia load there could be limitation in the position. The position can be limited by just incorporating a function that cuts off the variable value at the appropriate level. The situation is, however, complicated by the fact that there is a connection between speed and position. Therefore the speed has to be handled as well. It should take on the value of zero as soon as the position hits a limitation. Since the position is integrated from speed this situation leads to a dead-lock once a limitation is activated (since speed is zero as long as the position is limited, the position cannot start to move out of the limitation). Since introducing limitations on the variable x_2 in the form of x_{min} and x_{max} and also having it affect the derivative v_2 . Therefore, both speed and position are solved from essentially the same equation, thereby removing the dependency between them.

$$\mathbf{F} = \begin{pmatrix} M\dot{v}_2 + Bv_2 - f_1 + f_2 \\ M\dot{x}_2 + Bv_2 - f_1 + f_2 \\ v_1 + v_2 \\ x_1 + x_2 \\ -c_{x,1} - v_1 Z_{cx,1} + F_1 \\ -c_{x,2} - v_2 Z_{cx,2} + F_2 \end{pmatrix} \quad (3.50)$$

The limitations need to be introduced after transformation to time discrete form.

$$\mathbf{G} = \begin{pmatrix} DS(1, -hf_1 + hf_2 - 2 + Bhv_2Mv_2) - hf_1 + hf_2 + Bh^2v_2 + 2Mv_2 \\ DS(1, -2h^2f_1 + 2h^2f_2 + 2Bh^2v_2 - 8Mx_2) + \\ + DS(2, -h^2f_1 + h^2f_2 + Bh^2v_2 + 4Mx_2) \\ (-h^2f_1 + h^2f_2 + Bh^2v_2 + 4Mx_2) \\ v_1 + v_2 \\ x_1 + x_2 \\ -c_{x,1} - v_1 Z_{cx,1} + F_1 \\ -c_{x,2} - v_2 Z_{cx,2} + F_2 \end{pmatrix} \quad (3.51)$$

At this stage the limitations on x_2 can be introduced, using the *Limit2* function. (The *limit2* function is used instead of "Limit" is when not only the variable itself is affected, but there is also a derivative to be adjusted).

$$x_{2,L} = \text{limit2}(x_2, v_2, x_{min}, x_{max}) \quad (3.52)$$

The corresponding effect of the limitation on the speed becomes

$$v_{2,L} = v_2 \frac{\partial}{\partial x_2} (\text{limit2}(x_2, v_2, x_{min}, x_{max})) \quad (3.53)$$

It is hereafter renamed to

$$v_{2,L} = v_2 \text{limit2}'(x_2, v_2, x_{min}, x_{max}) \quad (3.54)$$

Here *limit2'* is the derivative of *limit2*. It is zero as long as the limitations in x are violated and v_2 is moving in the same direction as the constraint, but take on the value of one, in between.

The *Limit2* function is just like a normal limitation function

$$\text{limit2}(x, v, x_{min}, x_{max}) = \text{If}[x > x_{max}, x_{max}, \text{If}[x < x_{min}, x_{min}, x]] \quad (3.55)$$

but its derivative with respect to x is defined as:

$$\text{limit2}'(x, v, x_{min}, x_{max}) = \text{If}[x > x_{max} \wedge v > 0, 0, \text{If}[x < x_{min} \wedge v < 0, 0, 1]] \quad (3.56)$$

The *limit2* function is plotted as a function of x and v in Fig 13 and the expression used to represent *limit2'* is plotted in Fig 14. Here x_{max} is set to 1 and x_{min} to -1 .

From In[13]:=

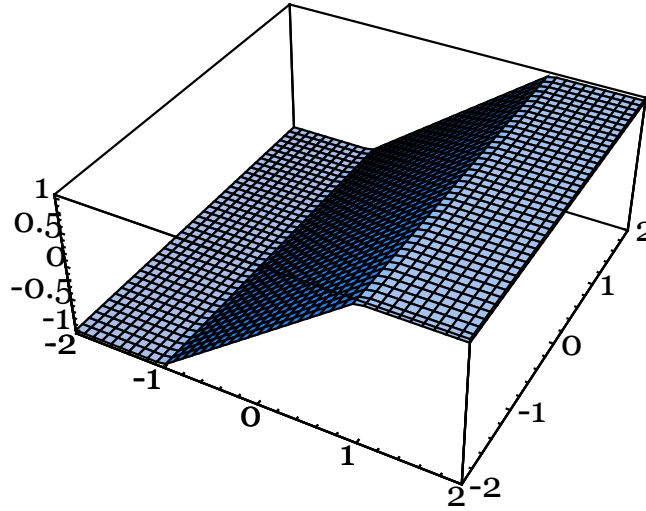


Figure 1

Figure 3.9: The *limit2* function as a function of x and v (v have o influence)

From In[14]:=

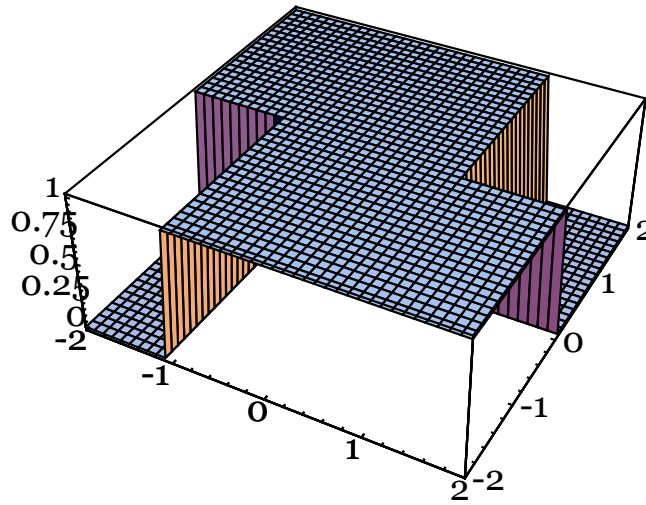


Figure 1

Figure 3.10: The *limit2'* function as a function of x and v

Using this limitation in the equation set yields

$$\mathbf{G} = \begin{pmatrix} v_2 + \frac{(DS(1, v_2(Bh-2M) + f_1(-h) + f_2h) + (f_2 - f_1)h) \text{Limit2}'(x_2, v_2, x_{\min}, x_{\max})}{Bh + 2M} \\ x_2 - \text{Limit2}\left(\frac{1}{2}(hv_2 - DS(1, -hv_2 - 2x_2)), v_2, x_{\min}, x_{\max}\right) \\ v_1 + v_2 \\ x_1 + x_2 \\ -c_{x,1} - v_1 Z_{cx,1} + F_1 \\ -c_{x,2} - v_2 Z_{cx,2} + F_2 \end{pmatrix} \quad (3.57)$$

This function can then be differentiated to obtain the Jacobian and be solved using Newton-Raphsson.

3.7 Modelling Two Masses as One Component

Sometimes it is useful to model two masses as one component. This could for instance be if there is a limitation on the relative position of these masses or if an actuator is inserted between the two masses, creating the need for a single node between the masses.

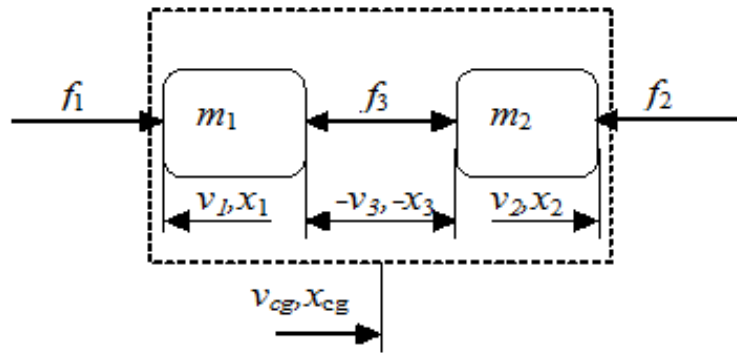


Figure 3.11: Two mass system modelled as one component.

The system of equations (without the boundary equations) for this system can be written as:

$$m_1 \dot{v}_1 = f_3 - f_1 \quad (3.58)$$

$$m_2 \dot{v}_2 = f_3 - f_2 \quad (3.59)$$

$$\dot{x}_1 = v_1 \quad (3.60)$$

$$\dot{x}_2 = v_2 \quad (3.61)$$

$$v_3 = v_1 + v_2 \quad (3.62)$$

$$x_3 = x_1 + x_2 \quad (3.63)$$

This is, however, not the alternative to do. If a limitation is to be introduced in x_3 (and v_3) it is more appropriate to use x_3 (and v_3) as one of the integration variables directly and to create another variable that is independent of x_3 as the other integration variable. The position of the centre of gravity for the two masses, can be calculated as:

$$(m_1 + m_2) s v_{cg} = m_2 s v_2 - m_1 s v_1 \quad (3.64)$$

Using that equation together with the equations for the speeds v_1 , v_2 , and v_3 yields the following system of equations

$$m_1 \dot{v}_1 = f_3 - f_1 \quad (3.65)$$

$$m_2 \dot{v}_2 = f_3 - f_2 \quad (3.66)$$

$$v_3 = v_1 + v_2 \quad (3.67)$$

$$(m_1 + m_2) \dot{v}_{cg} = m_2 \dot{v}_2 - m_1 \dot{v}_1 \quad (3.68)$$

This system of equations can be solved for v_3 and v_{cg} (and v_1, v_2) to yield the following equations

$$\dot{v}_3 = -\frac{f_2 m_1 - f_3 m_1 - f_1 m_2 - f_3 m_2}{m_1 m_2} \quad (3.69)$$

$$\dot{v}_{cg} = \frac{f_1 - f_2}{m_1 + m_2} \quad (3.70)$$

v_1 and v_2 can then be expressed as functions of these as:

$$v_1 = -\frac{m_1 v_{cg} + m_2 v_{cg} - m_2 v_3}{m_1 + m_2} \quad (3.71)$$

$$v_2 = -\frac{-m_1 v_{cg} - m_2 v_{cg} - m_1 v_3}{m_1 + m_2} \quad (3.72)$$

The new total system of equations for this system, where also the boundary conditions has been added can be written as:

$$\dot{v}_{cg} = -\frac{f_2 m_1 - f_3 m_1 - f_1 m_2 - f_3 m_2}{m_1 m_2} \quad (3.73)$$

$$\dot{v}_3 = \frac{f_1 - f_2}{m_1 + m_2} \quad (3.74)$$

$$\dot{x}_{cg} = x_{cg} \quad (3.75)$$

$$\dot{x}_3 = x_3 \quad (3.76)$$

$$v_1 = -\frac{m_1 v_{cg} + m_2 v_{cg} - m_2 v_3}{m_1 + m_2} \quad (3.77)$$

$$v_2 = -\frac{-m_1 v_{cg} - m_2 v_{cg} - m_1 v_3}{m_1 + m_2} \quad (3.78)$$

$$x_2 = -\frac{-m_1 x_{cg} - m_2 x_{cg} - m_1 x_3}{m_1 + m_2} \quad (3.79)$$

$$x_1 = x_3 - x_2 \quad (3.80)$$

$$f_1 = c_{x,1} + v_1 Z_{cx,1} \quad (3.81)$$

$$f_2 = c_{x,2} + v_2 Z_{cx,2} \quad (3.82)$$

$$f_3 = c_{x,3} + v_3 Z_{cx,3} \quad (3.83)$$

In this way, the limitations, or actuator, can be placed and affect only x_3 and v_3 directly.

3.8 A Mechanical Spring as a Transmission line element

Modelling of a mechanical spring as a transmission line element is similar to the modelling of a hydraulic capacitance, but with speed instead of flow and force instead of pressure. Note that the

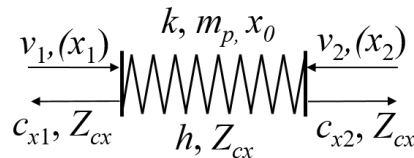


Figure 3.12: Spring as a unit transmission line element

sign convention is here the same as for a fluid system. It could, however, be argued that a direction should be introduced, since it is obvious to do so for the two and three dimensional cases, where position, and force etc are vectors in space. As for now in the One dimensional case they are treated as scalars.

$$c_{x,2}(t+h) = f_1(t) + Z_{cx}v_1(t) \quad (3.84)$$

$$c_{x,1}(t+h) = f_2(t) + Z_{cx}v_2(t) \quad (3.85)$$

Inserted in (31) and (32) yields

$$f_1(t) = c_{x,1}(t) = +Z_{cx}v_1(t) \quad (3.86)$$

$$f_2(t) = c_{x,2}(t) = +Z_{cx}v_2(t) \quad (3.87)$$

where c_1 and c_2 contain all information of the opposite node necessary for the calculation of pressure and flow at one node. The characteristic impedance is calculated:

$$Z_{cx} = \frac{h}{k} \quad (3.88)$$

At each component between the lines, we then have to solve the following system of equations.

$$v = v(f) \quad (3.89)$$

$$f = c_x + vZ_{cx} \quad (3.90)$$

The parasitic impedance, which here has the unit of mass, and hence represents a parasitic mass, can be calculated as

$$m_p = hZ_{cx} = \frac{h^2}{k} \quad (3.91)$$

This means that the spring model also has a mass distributed along its length. This of course represents a modelling error if an ideal spring should be modelled. If the spring is connected to a mass m_L this will reduce the resonant frequency of the system. However, since the parasitic mass can be computed the mass that it is connected to, can be reduced correspondingly. The reduced mass can be calculated as:

$$m'_L = m_L - \frac{m_p}{2} \quad (3.92)$$

This can of course only go on as long as the reduced mass is greater than zero.

Since the primary variables in the spring is the speeds and these speeds are integrated into positions at the adjacent component there is a risk that there is a numerical drift in the integration represented by the transmission line component, with respect to the integrations in the adjacent component. Therefore it can be useful to have a weak feedback to adjust the values of f_1 and f_2 so that the average becomes:

$$f = k(x_2 - x_1 - x_0) \quad (3.93)$$

Here x_0 is the free length of the spring. Introducing the force error as:

$$f_e = f - \frac{1}{2}(f_1 + f_2) \quad (3.94)$$

The transmission line equations is then modified to

$$c_{x,2}(t+h) = f_1(t) + Z_{cx}v_1(t) + f_e \epsilon_e \quad (3.95)$$

$$c_{x,1}(t+h) = f_2(t) + Z_{cx}v_2(t) + f_e \epsilon_e \quad (3.96)$$

where ϵ_e is a very small number. In this way numerical drift can be avoided.

3.9 Rotational Mechanical System

In rotational mechanical systems we are looking at rotating shafts and machines. This is particularly useful to represent drive trains. The most fundamental components here are a moment of inertia (instead of a mass) and a rotating shaft instead of a spring. In addition, rotating system almost always contains gears.

3.10 A General Formulation of a Mechanical System

In order to make a realistic simulation of a hydraulic system it is necessary to include a representation of the load. Often this is a complex mechanical structure with interacting parts. This is true in e.g. mobile systems, such as lorry cranes, forest machines and excavators. Fig. 3.13 shows a simplified model of a crane represented by just three degrees of freedom. It would be possible to use a more complete 3D models to have more accurate description of the system dynamics. However, in many cases, and especially in conceptual design, a 2D model is just as adequate. The computational burden and also the modelling work, increases steeply with a full 3D model. Even if e.g. a crane can move in three dimensions the dimensions have a weak coupling. E.g. the rotation is only weakly coupled to the movements of the pistons. Computational speed is particularly important in for conceptual design when simulation based optimisation is used. Here thousands of simulations might be needed, and therefore, simulation speeds much faster than real time is required. Especially when designing a e.g. a hydraulic control system, the mechanical model need not have much more degrees of freedoms than the number of actuator. Typically the flexibility in a mechanical structure can be aggregated at the base as the stiffness k_0 in Fig. 3.13.

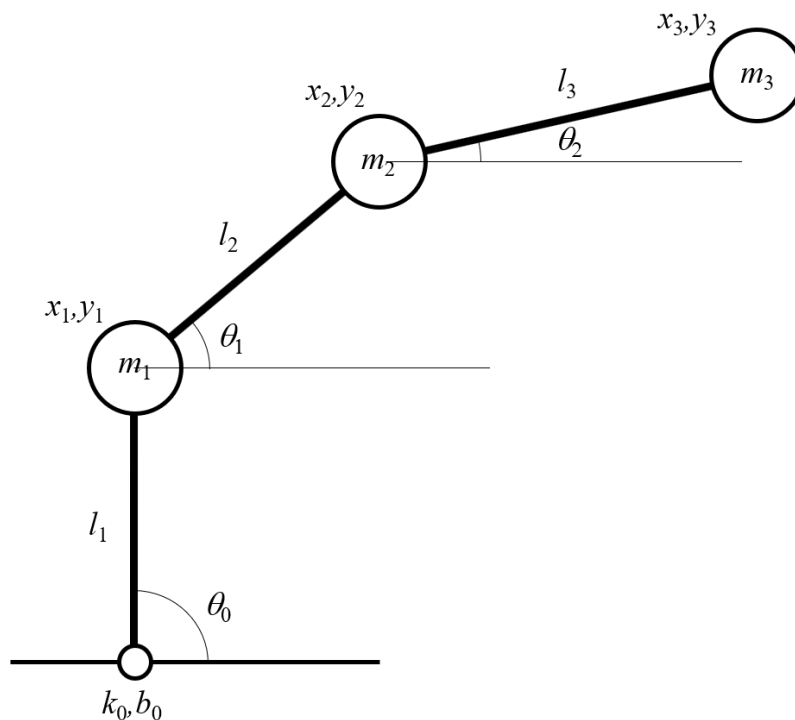


Figure 3.13: Simplified representation of a crane with three degrees of freedoms. There is a flexible joint at the base and two actuated joints. There is one point masses for each joint and one at the tip. They are connected with stiff beams

A flexible joint is located at the base of the crane to represent the flexibility inherent in the

structure. Each beam is represented by a mass and an inertia moment.

As a consequence the crane can be described by three coordinates corresponding to the angles θ that are used as the independent generalized coordinates. This is indicated in Fig. 3.13.

3.10.1 The Euler-Lagrange equation

One approach that is useful for this kind of mechanics is the extended Euler-Lagrange equation.

$$\frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{\Theta}} - \frac{\partial L}{\partial \Theta} + \frac{\partial D}{\partial \dot{\Theta}} = F_{\Theta} \quad (3.97)$$

Here L is the difference between kinetic energy T and potential energy V .

$$L = T - V \quad (3.98)$$

It has here been extended with the dissipative power D .

Considering a simple mass spring damper system.

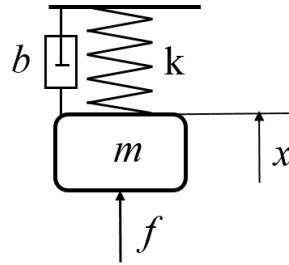


Figure 3.14: Mass spring damper system.

For this system these terms would be:

$$T = m \frac{\dot{x}^2}{2} \quad (3.99)$$

$$V = k \frac{x^2}{2} + mgx \quad (3.100)$$

Hence

$$L = m \frac{\dot{x}^2}{2} - k \frac{x^2}{2} - mgx \quad (3.101)$$

$$D = b \frac{\dot{x}^2}{2} \quad (3.102)$$

Furthermore $F_{\Theta} = f$ and with $\Theta = x$, Eq. 3.97 becomes:

$$\frac{\partial}{\partial t} \frac{\partial}{\partial \dot{x}} \left(m \frac{\dot{x}^2}{2} - k \frac{x^2}{2} - mgx \right) - \frac{\partial}{\partial x} \left(m \frac{\dot{x}^2}{2} - k \frac{x^2}{2} - mgx \right) + \frac{\partial}{\partial \dot{x}} \left(b \frac{\dot{x}^2}{2} \right) = f \quad (3.103)$$

Performing the differentiations yields:

$$m\ddot{x} + b\dot{x} + kx + mg = f \quad (3.104)$$

In this way we have arrived to the well known equation of motion for this spring-mass system. Of course, for this simple system, this is somewhat of a detour, but for more complicated systems it is a rather powerful method.

The elements for Lagrange equation for the crane system in Fig. 3.13 can be written as:

$$T = m_1 \frac{\dot{x}_1^2 + \dot{y}_1^2}{2} + m_2 \frac{\dot{x}_2^2 + \dot{y}_2^2}{2} + m_3 \frac{\dot{x}_3^2 + \dot{y}_3^2}{2} \quad (3.105)$$

The potential energy is:

$$V = k_0 \frac{\theta_0 - \theta_{base}}{2} + m_1 g y_1 + m_2 g y_3 + m_3 g y_3 \quad (3.106)$$

Having some viscous friction at the joints would generate dissipative power.

$$D = b_{v0} \frac{\dot{\theta}_0}{2} + b_{v1} \frac{\dot{\theta}_1}{2} + b_{v2} \frac{\dot{\theta}_2}{2} \quad (3.107)$$

Furthermore, the geometric relations have to be established. Using the angles and their differentials as the independent variables, the positions and their derivatives can be calculated as:

$$x_1 = l_1 \cos \theta_0 \quad (3.108)$$

$$y_1 = l_1 \sin \theta_0 \quad (3.109)$$

$$x_2 = l_2 \cos \theta_1 + x_1 \quad (3.110)$$

$$y_2 = l_2 \sin \theta_1 + y_1 \quad (3.111)$$

$$x_3 = l_3 \cos \theta_2 + x_2 \quad (3.112)$$

$$y_3 = l_3 \sin \theta_2 + y_2 \quad (3.113)$$

Then the relationships for the differentials are simply

$$\dot{x}_1 = \frac{d}{dt}(l_1 \cos \theta_0) = -l_1 \dot{\theta}_0 \sin \theta_0 \quad (3.114)$$

and so on for the other variables. If there are angular motors at each joint, the moment from these would be inserted into the vector F_Θ . However, if pistons are used the geometry needs to be defined. In Fig. 3.15 a general interface is shown. Since the interface to this model would be the pistons, the boundary equations have to be expressed in the piston coordinates.

$$f_{pi} = c_{pi} + Z_{cpi} \dot{x}_{pi} \quad (3.115)$$

The speed at the pistons are:

$$\dot{x}_{pi} = -r_i (\dot{\theta}_i - \dot{\theta}_{i-1}) \quad (3.116)$$

The moments for each joint can be calculated from the piston forces as:

$$T_i = f_{pi} r_i \quad (3.117)$$

To get the piston position and the radius r_i can be obtained from the figure 3.15 as The total length of the cylinder is:

$$l_{pi} = \sqrt{(x_{ci2} - x_{ci1})^2 + (y_{ci2} - y_{ci1})^2} \quad (3.118)$$

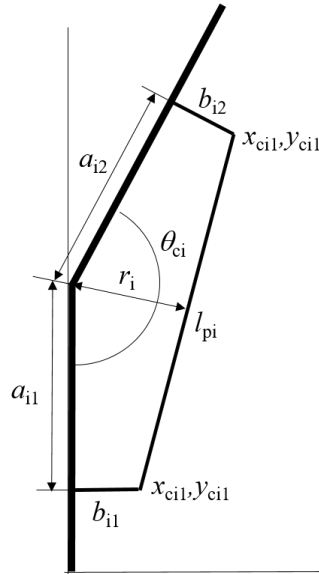


Figure 3.15: Interface of a piston to a mechanical joint.

Here

$$x_{ci1} = b_{i1} \quad (3.119)$$

$$y_{ci1} = -a_{i1} \quad (3.120)$$

$$x_{ci2} = a_{i2} \sin \theta_{ci} + b_{i2} \cos \theta_{ci} \quad (3.121)$$

$$y_{ci2} = -a_{i2} \cos \theta_{ci} - b_{i2} \sin \theta_{ci} \quad (3.122)$$

The instantaneous radius can be calculated as:

$$r_i = \frac{dl_{p1}}{d\theta_{ci}} \quad (3.123)$$

where

$$\theta_{ci} = \pi + \theta_i - \theta_{i-1} \quad (3.124)$$

Here

$$\theta_{ci} = \pi + \theta_i - \theta_{i-1} \quad (3.125)$$

Finally

$$x_{pi} = -(l_{p1} - x_{0i}) \quad (3.126)$$

Here x_0 is the length of the totally compressed piston. The minus sign is because of the sign convention. The piston has a negative speed going out and hence the piston position is always negative.

3.10.2 General formulation of mechanical structures for TLM

A more general formulation of the mechanical structure is represented by equation.

$$\mathbf{M}(\Theta)\ddot{\Theta} + \mathbf{B}(\Theta, \dot{\Theta})\dot{\Theta} + \mathbf{K}\Theta + \mathbf{F}_{g,\Theta} = \mathbf{F}_{\Theta} \quad (3.127)$$

where Θ is a position/angle vector and \mathbf{F}_Θ is a force/moment vector of external forces eg actuators. $\mathbf{F}_{g,\Theta}$ is a vector representing gravitational forces. \mathbf{M} is the inertia matrix, \mathbf{B} is the matrix representing friction, centripetal and Coriolis effects, and \mathbf{K} is the stiffness matrix. These matrices could be the outputs from some FEM-program or from some program that linearises the non-linear dynamic equations of a mechanical system. The force vector from the actuators can be written

$$\mathbf{F}_x = \mathbf{c}_x + \dot{\mathbf{x}}\mathbf{Z}_{cx} \quad (3.128)$$

where \mathbf{c}_x is a vector with force characteristics, \mathbf{x} is a vector with the actuator positions and \mathbf{Z}_{cx} is a diagonal matrix with the characteristic impedances of the actuators in the diagonal. This equation is similar to (3.86) except that it is on vector form.

Before they are able to interact with the mechanical structure they have to be transformed into the coordinates of the structure. The speeds of the mechanical structure are transformed into speeds at the actuators.

$$\dot{\mathbf{x}} = -\mathbf{R}(\Theta)\dot{\Theta} \quad (3.129)$$

The negative sign is due to the fact that the direction of force and resulting motion is the same in (3.127) but opposite in (3.89). The forces from the actuators are transformed into forces/moments acting on the mechanical structure by the same matrix \mathbf{R} . All dependency of $\dot{\Theta}$ on \mathbf{R} is hereafter assumed implicitly.

$$\mathbf{F}_\Theta = \mathbf{R}^T \mathbf{F}_x \quad (3.130)$$

Using (3.128) and (3.129) in (3.130) the force/moment vector can be written

$$\mathbf{F}_\Theta = \mathbf{R}^T (\mathbf{c}_x - \mathbf{Z}_{cx}\mathbf{R}\dot{\Theta}) \quad (3.131)$$

Introducing

$$\mathbf{c}_\Theta = \mathbf{R}^T \mathbf{c}_x \quad (3.132)$$

The characteristic impedance projected on the angular coordinates becomes:

$$\mathbf{Z}_\Theta = \mathbf{R}^T \mathbf{Z}_{cx} \mathbf{R} \quad (3.133)$$

Equation (3.131) can then be rewritten as

$$\mathbf{F}_\Theta = \mathbf{c}_\Theta - \dot{\Theta}\mathbf{Z}_{c,\Theta} \quad (3.134)$$

Equation (3.134) in (3.127) and rearranging yields

$$\ddot{\Theta} = -\mathbf{M}^{-1}(\mathbf{B} + \mathbf{Z}_{c,\Theta} + \mathbf{B}) - \mathbf{K}\Theta + \mathbf{F}_{g,\Theta} + \mathbf{c}_\Theta \quad (3.135)$$

In (3.135), \mathbf{c}_Θ only contains old information from the previous time step. This means that it can be regarded as an independent input signal. In (3.134) the input signal \mathbf{F} is strongly dependent on $\dot{\Theta}$, which means that Θ cannot be solved independently from \mathbf{F} . Equation (3.135) can be written on state space form

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{u} \quad (3.136)$$

where

$$\mathbf{A} = \begin{pmatrix} -\mathbf{M}^{-1}(\mathbf{Z}_\Theta + \mathbf{B}) & -\mathbf{M}^{-1}\mathbf{K} \\ \mathbf{I} & 0 \end{pmatrix} \quad (3.137)$$

$$\mathbf{u} = \begin{pmatrix} \mathbf{F}_{g,\Theta} + \mathbf{c}_\Theta \\ 0 \end{pmatrix} \quad (3.138)$$

$$\mathbf{y} = \begin{pmatrix} \dot{\Theta} \\ \Theta \end{pmatrix} \quad (3.139)$$

This equation can then be solved using an implicit integration method such as the trapez method, since everything is known and the Jacobian can be easily computed. The interface are through the wave variables \mathbf{c}_θ associated with force, and these can be regarded as pure inputs. This demonstrate one of the main advantages of the transmission line modelling, that it can be used to bridge different kinds of systems together. in this case the mechanical structure can be solved independently from other components (in each time step) since it is numerically isolated from the other.

3.11 Electrical Components

Electrical components can be modelled in much the same way as hydraulic components. In fact the term transmission line modelling TLM was coined in the paper by Johns and O'Brien [7] where it was applied to electrical circuits.

Although there are large conceptual similarities between electrical and hydraulic systems, there are also important differences. One difference is that electrical system in general are more stiff. This basically come down from the fact that the signal propagation speed in electrical system is very high, i.e. the speed of light wich is 3×10^8 m/s, which is more that five orders of magnitude higher than in hydraulic systems. Another difference is that electrical components are in general more linear than their hydraulic counterparts. This means that differential algebraic solvers can be used efficiently also for larger systems. However, for large systems it is still very usefull to use transmission line elements to partition a model. Furthermore, depending on the time scale of interesst electrical systems can be simulated using transmission line method in its entierly. This is particularly true when it comes to model electrical power systemsto be used in conceptual design.

3.11.1 Electrical capacitance

The two port electrical capacitance is the equivalent of a two port hydraulic volume.

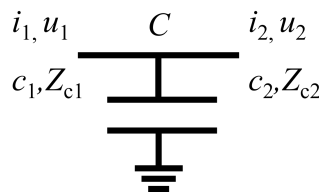


Figure 3.16: Two port electrical capacitance

$$c_1(t) = Z_c i_2(t - T) + u_2(t - T) \quad (3.140)$$

$$c_2(t) = Z_c i_1(t - T) + u_1(t - T) \quad (3.141)$$

c_1, c_2 are called the characteristics and they represents the waves traveling in each direction through the line. Z_c is the characteristic impedance and is calculated from the capacitance C and the time step h .

$$Z_c = h/C \quad (3.142)$$

The parasitic inductance is:

$$L_{parasitic} = Z_c h = \frac{h^2}{C} \quad (3.143)$$

Filtering of the wave variables can be introduced in the same way as for the hydraulic capacitance. It is also possible to implement a multi port version in the same way as for hydraulic volumes.

3.11.2 Electrical inductance

Interestingly an electrical inductance can be modelled both as a Q-component and as a C-component. When modelled as a C-component i.e. as a transmission line element the characterist

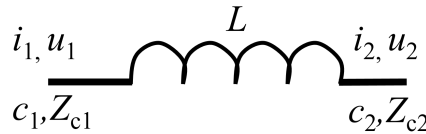


Figure 3.17: Electrical inductance

impedance is chosen such that the inductance is correct.

$$Z_c = L/h \quad (3.144)$$

This means there will be a parasitic capacitance.

$$C_{parasitic} = h/Z_c = \frac{h^2}{L} \quad (3.145)$$

Modelling the inductance as a Q-component means that the following system of differential algebraic equations has to be solved:

$$\frac{di_2}{dt} = (u_1 - u_2)/L \quad (3.146)$$

$$i_1 = -i_2 \quad (3.147)$$

$$u_1 = c_1 + Z_{c1}i_1 \quad (3.148)$$

$$u_2 = c_2 + Z_{c2}i_2 \quad (3.149)$$

3.11.3 Electrical resistance

The electrical resistance is the equivalent of a hydraulic laminar orifice.

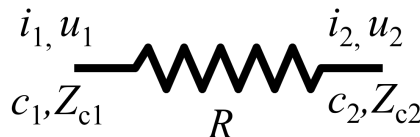


Figure 3.18: Electrical resistance

When the restrictor is connected to transmission lines the following equations are solved for

the restrictor.

$$i_2 = (u_1 - u_2)/R \quad (3.150)$$

$$i_1 = -i_2 \quad (3.151)$$

$$u_1 = c_1 + Z_{c1}i_1 \quad (3.152)$$

$$u_2 = c_2 + Z_{c2}i_2 \quad (3.153)$$

The equations used in the component will thus be

$$i_2 = \frac{c_1 - c_2}{R + Z_{c1} + Z_{c2}} \quad (3.154)$$

$$i_1 = -i_2 \quad (3.155)$$

$$u_1 = c_1 + Z_{c1}i_1 \quad (3.156)$$

$$u_2 = c_2 + Z_{c2}i_2 \quad (3.157)$$



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